Ray-Triangle and Ray-Quadrilateral Intersections

in

Homogeneous Coordinates

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Experience shows that operation with homogeneous coordinates produces, at least with beginners, something like physical discomfirt. Felix Klein

Recent articles in *Ray Tracing News* have discussed solutions to the problem of intersecting a ray with a triangle using the triangle's barycentric coordinates. This article shows yet another way to think about the ray-triangle intersection problem. The idea is to think of the barycentric coordinates of the intersection point, not as the ratio of areas, but rather as the as ratios of the volume of tetrahedra. This technique can also be used to derive formulae for ray-quadrilateral intersection by using the fact that associated with every planar quadrilateral is a unique triangle. Finally, I would like to show the connection between these results and Plucker line coordinates. This insight leads to quite an efficient algorithm.

Most people studying computer graphics learn about homogeneous coordinates and projective geometry because they allow perspective transformations to be expressed as linear transformations involving 4x4 matrices. When writing a ray tracer, however, most authors have duplicated Turner Whitted's original approach where eye rays are directly formed in world space, obviating the need for a explicit viewing transformation. As a result the geometric calculations are usually expressed using familiar vector algebra in a Euclidean coordinate system. However, homogeneous coordinates were not invented to just express projective transformations; they have the advantage that many geometric calculations have elegant solutions when expressed in term of these coordinates. In fact, Felix Klein goes on to say in the same book from which the opening quote is taken.

> *For our general standpoint, the questions of ordinary vector analysis constitute only a chapter out of a profusion of more general problems [in geometry].*

For those interested in more about homogeneous coordinates and their uses in geometry, I suggest the references at the end of the article and I particular a recent thesis by J. Stolfi. I also might recommend my paper entitled, "The Homogeneous Geometry Calculator," which describes how these ideas can be used in geometric modeling and computer graphics.

Points, Planes, Lines and Determinants

In three-dimensional projective geometry the homogeneous coordinates of points are 4-vectors

$$\mathbf{P} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

The 3D location of a point is interpreted to be $(x/w, y/w, z/w)$. Because of the division by w, the homogeneous coordinates of a point can be multiplied by a constant (other than 0) without effecting its location. If w is equal to 0, the point is interpreted to at *infinity* in the direction given by the first three coordinates. In projective geometry, points at infinity in opposite directions are considered the same.

A line is uniquely determined by two points. All points on the line can be generated by forming linear combinations of two points (Figure 1).

$$\mathbf{P} = s_1\mathbf{P1} + s_2\mathbf{P2}$$

One can interpret this equation by stating that **P1** and **P2** define a coordinate system on the line, and that the coordinates of an individual point are are (s_1, s_2). In particular, the coordinates of **P1** are $(1, 0)$ and the coordinates of **P2** are $(0, 1)$. Notice as s_1 increases positively that **P** moves towards **P1** and as s_2 increases positively **P** moves towards **P2**. Furthermore, these coordinates have a nice geometric interpretation: s_1 is the ratio of the length of **PP1** to the length of **P1P2**, and s_2 is the ratio of the length of **P P2** to the length of **P1P2**, These observations allow us to enumerate the set of points on the line segment **P1P2**.

$$\{\, (s_1, s_2) : (s_1 > 0)\text{and}(s_2 > 0) \,\}$$

Often it is convenient to normalize the coordinates so that their sum is one.

$$s_1\prime = \frac{s_1}{s_1 + s_2}$$

$$s_2\prime = \frac{s_2}{s_1 + s_2}$$

Coordinates normalized in this way are referred to as *barycentric coordinates* and were originally developed by Moebius. With this representation, if the two points **P1** and **P2** both have $w = 1$, then all linear combinations involving barycentric coordinates will also have $w = 1$. The advantage of this from a computational point of view is that this allows us to effectively ignore the extra coordinate. However, if we allow other values for w, it is not necessary to normalize the coordinates in this way, in the process saving the cost of performing the division and avoiding a nasty singularity if the denominator happens to be 0. A basic rule of thumb when using homogeneous coordinates is: Be suspicious of any formula requiring a division. Said another way: Delay all divisions until it is absolutely necessary to interpret a point as a 3D location.

Unfortunately, there is a subtlety having to do with signs. As shown above, we defined the interval between **P1** and **P2** as being the region where s_1 and s_2 are both positive.

Now suppose s_1 and s_2 are both multiplied by -1. Then they are both negative, and the point formed as their linear combination -1 times the point they originally generated. But, by our convention that multiplying a point by a constant does not change its position, these two points represent the same location on the line. Note, however, in this case the point generated will $w < 0$. It is tempting then to divide homogeneous points into two classes based on the sign of w, and to say that a point with negative w coordinate is inside other points with negative w coordinates (since it can then be formed by positive linear combinations of them). A thorough discussion of this approach is contained in J. Stolfi's thesis. For the purposes of this paper, we will always assume $w > 0$. Without this assumption all the formulas involving signs, orientations and determinants are not valid.

In the same way that an axiom of projective geometry states that a unique line is determined by two points, so another axiom states that a unique plane is determined by three points. All points on this plane can be generated by forming linear combinations of three points (Figure 2).

$$\mathbf{P} = u_1\mathbf{P1} + u_2\mathbf{P2} + u_3\mathbf{P3}$$

The coordinates of point in the plane coordinate system defined by the triangle **P1P2P3** are (u_1, u_2, u_3). These coordinates have the geometric interpretation that u_1 is the ratio of the area of triangle **PP2P3** to **P1P2P3**, and u_2 is the ratio of **P1PP3** to **P1P2P3**, and u_3 is the ratio of **P1P2P** to **P1P2P3**. The set of points inside the triangle is

$$\{\, (u_1, u_2, u_3) : (u_1 > 0)\text{and}(u_2 > 0)\text{and}(u_3 > 0) \,\}$$

Now suppose there is a fourth point **P4** contained in the plane. Then **P4** can be written as a linear combination of **P1**, **P2**, and **P3**. So there must be some set of values (u_1, u_2, u_3, u_4) which satisfy the following equation

$$u_1\mathbf{P1} + u_2\mathbf{P2} + u_3\mathbf{P3} + u_4\mathbf{P4} = 0$$

Let's write this out as a set of linear equations

$$\begin{aligned} x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 &= 0 \\ y_1u_1 + y_2u_2 + y_3u_3 + y_4u_4 &= 0 \\ z_1u_1 + z_2u_2 + z_3u_3 + z_4u_4 &= 0 \\ w_1u_1 + w_2u_2 + w_3u_3 + w_4u_4 &= 0 \end{aligned}$$

Recall from linear algebra that this set of homogeneous equations will have a solution iff the following determinant equals 0.

$$|\mathbf{P1P2P3P4}| = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} = 0$$

If we interpret **P4** as an arbitrary point $\mathbf{P} = (x, y, z, w)$ in the plane, this equation is the plane equation.

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} x - \begin{vmatrix} x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} y + \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ w_1 & w_2 & w_3 \end{vmatrix} z - \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} w = 0$$

which can be derived by expanding the 4x4 determinant into subdeterminants. The four 3x3 subdeterminants form the homogeneous coordinates of the plane through the three points. We will symbolically express this equation as

$$|\mathbf{P1P2P3}| \cdot \mathbf{P} = 0$$

Although we will not go into it in this article, all the results and theorems in projective geometry are symmetric with respect to points and planes. This means that any result involving points may be interpreted as a result involving planes just by replacing the word point with planes. This is called *duality*. This implies that all the calculations performed with homogeneous point coordinates have dual analogues using homogeneous plane coordinates.

Suppose that **P4** is not in the plane. Then the four points determine a coordinate system in space. All points in space can be written as linear combinations of these 4 points (Figure 3).

$$\mathbf{P} = u_1\mathbf{P1} + u_2\mathbf{P2} + u_3\mathbf{P3} + u_4\mathbf{P4}$$

The barycentric coordinates in a tetrahedron can be interpreted as the ratio of volumes of tetrahedra. That is, u_1 is the ratio of the volume of the tetrahedra **PP2P3P4** to **P1P2P3P4**, u_2 is the ratio of **PP3P4P1** to **P1P2P3P4**, etc.

The final geometric fact that we need is that the determinant of four points is proportional to the volume of the tetrahedron defined by the four points.

$$volume(\mathbf{P1P2P3P4}) = 1 over 6 \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}$$

(If the w coordinates of the four points are not 1's, the volume is multiplied by the product of the four w coordinates.) The verification of this formula is left as an exercise. It's also useful to recall two other properties of determinants. First, the sign of a determinant changes if adjacent rows or columns are interchanged.

$$|\mathbf{P1P2P3P4}| = -|\mathbf{P2P1P3P4}| = |\mathbf{P2P3P1P4}| = \text{...etc.}$$

Thus the volume of the tetrahedra is also signed. This can be interpreted geometrically by saying that the volume $|\mathbf{P1P2P3P4}|$ is positive if it is *right-handed*. It is right-handed if when the fingers of your right hand are made to follow **P2P3P4**, your thumb points towards **P1**. Second, if any point is duplicated in a determinant, the determinant is equal to 0; this is intuitively obvious given the interpretation of the determinant as a volume.

Ray-Triangle Intersection

With the above preliminaries, we are now prepared to write down the ray-triangle intersection formulas. Define a ray by two points **R1** and **R2**

$$\mathbf{R} = s_1\mathbf{R1} + s_2\mathbf{R2},$$

and the triangle by three points **P1**, **P2**, and **P3**

$$\mathbf{P} = u_1\mathbf{P1} + u_2\mathbf{P2} + u_3\mathbf{P3}.$$

This situation is illustrated in Figure 4.

To solve for (s_1, s_2) observe that the point of intersection must lie on the line containing the ray **R1R2** and on plane containing the triangle **P1P2P3**. These constraints are expressed by

$$|\mathbf{P1P2P3}| \cdot \mathbf{R} = s_1|\mathbf{P1P2P3R1}| + s_2|\mathbf{P1P2P3R2}| = 0$$

By inspection, this equation has a solution at

$$\begin{aligned} s_1 &= |\mathbf{P1P2P3R2}| \\ s_2 &= |\mathbf{P3P2P1R1}| \end{aligned},$$

which can be verified as follows:

$$\begin{aligned} &|\mathbf{P1P2P3R2}||\mathbf{P1P2P3R1}| + |\mathbf{P3P2P1R1}||\mathbf{P1P2P3R2}| = \\ &|\mathbf{P1P2P3R2}||\mathbf{P1P2P3R1}| - |\mathbf{P1P2P3R1}||\mathbf{P1P2P3R2}| = 0. \end{aligned}$$

The geometric interpretation of this solution is that s_1 is equal to the volume of the tetrahedron **P1P2P3R2**, and s_2 is equal to the volume of the tetrahedron **P3P2P1R1**. This is illustrated in Figure 4A.

Alternatively, to solve for (u_1, u_2, u_3) observe that the point of intersection must also lie in any plane containing the ray. In particular it must lie in the planes of the triangles **R1R2P1**, **R1R2P2**, and **R1R2P3**.

$$|\mathbf{R1R2P1}| \cdot \mathbf{P} = u_2|\mathbf{R1R2P1P2}| + u_3|\mathbf{R1R2P1P3}| = 0$$

$$|\mathbf{R1R2P2}| \cdot \mathbf{P} = u_1|\mathbf{R1R2P2P1}| + u_3|\mathbf{R1R2P2P3}| = 0$$

$$|\mathbf{R1R2P3}| \cdot \mathbf{P} = u_1|\mathbf{R1R2P3P1}| + u_2|\mathbf{R1R2P3P2}| = 0$$

By inspection, these equations have solutions

$$\begin{aligned} u_1 &= |\mathbf{R1R2P2P3}| \\ u_2 &= |\mathbf{R1R2P3P1}| \\ u_3 &= |\mathbf{R1R2P1P2}| \end{aligned}$$

Verifying this for the first case yields:

$$\begin{aligned}
u_2|\mathbf{R1R2P1P2}| + u_3|\mathbf{R1R2P1P3}|& \\
&= |\mathbf{R1R2P3P1}||\mathbf{R1R2P1P2}| + |\mathbf{R1R2P1P2}||\mathbf{R1R2P1P3}| \\
&= |\mathbf{R1R2P3P1}||\mathbf{R1R2P1P2}| - |\mathbf{R1R2P1P2}||\mathbf{R1R2P3P1}| = 0
\end{aligned}$$

The other cases are just as easily verified. These solutions also have a very simple geometric interpretation. The coordinate u_1 is equal to the volume of the tetrahedron **R1R2P2P3**. In fact, all the coordinates are volume of tetrahedra formed by the two end points of the ray and the two points opposite the point associated with the coordinate. This is shown in Figure 4B.

To actually test for an intersection with the triangle we need to determine whether the intersection is inside the triangle. This is easily done by testing whether all three coordinates are positive. In fact, it is best to test each coordinate as it is computed, since if it is negative, it is not necessary to compute the other coordinates. To test whether the intersection point is contained between the two endpoints of the rays, check that both s_1 and s_2 are positive.

Ray-Quadrilateral Intersection

To test whether a planar quadrilateral is intersected by a ray can be done using the above triangle intersection algorithm. This is possible because every planar quadrilateral defines a unique reference triangle. The coordinates of the vertices of the quadrilateral in this reference triangle are: (1,1,1), (-1,1,1), (1,-1,1) and (-1,-1,1). And, the equations of lines connecting the four sides are:

$$\begin{aligned}
u_1 + u_3 &>= 0 \\
-u_1 + u_3 &>= 0 \\
u_2 + u_3 &>= 0 \\
-u_2 + u_3 &>= 0
\end{aligned}$$

So, a ray-quadrilateral intersection can be computed by first computing the coordinates of the ray intersection with the reference triangle as described in the last section. Then, these coordinates are tested to see whether they lie inside the unit square.

Figure 5 shows the reference triangle associated with quadrilateral whose corners are **A**, **B**, **C** and **D**. These points naturally define six lines: two pairs of opposites, (**AB**, **CD**) and (**AD**, **BC**); and a diagonal pair, (**AC**, **BD**). These pairs intersect in three points which define a triangle: $\mathbf{P1} = \mathbf{AB} \times \mathbf{CD}$, $\mathbf{P2} = \mathbf{AD} \times \mathbf{BC}$, and $\mathbf{P3} = \mathbf{AC} \times \mathbf{BD}$. Note that if two lines in a line pair are parallel, they intersect at a point at infinity. The proof that in this coordinate system the quadrilateral is a unit square can be found in the references. [Texture mapping.]

Line Coordinates

Finally let me mention a connection between these determinants and another representation of lines in homogeneous coordinates. To motivate this new representation, note that the ray-triangle calculation involves calculating determinants of the type $|\mathbf{P1P2R1R2}|$. Normally when generating an image in a ray tracer, each triangle is tested against many rays and each ray against many triangles. Thus, it is worthwhile to preprocess triangles and rays since the preprocessing cost will be amortized over all the intersection tests. So this leads to the question: Is there some way to rearrange the above determinant into a set of terms involving only **P1** and **P2** and another set only involving **R1** and **R2**? If this is done, then these terms can be precalculated. An example of a similar rearrangement of determinants was the plane $|\mathbf{P1P2P3}|$. This is an interpreted as four 3 subdeterminants which can be precomputed. To compute the 4x4 determinant **P1P2P3R2**, all that is required is a four element dot product of the plane **P1P2P3** with the point **R2**.

In the line representation by two points used above, the coordinates of points on the line depend on the choice of **R1** and **R2**. Is there a line representation that does not change if **R1** and **R2** change? One such choice are the 2x2 determinants

$$p_x y = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

$$p_x z = \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} = x_1 z_2 - x_2 z_1$$

$$p_x w = \begin{vmatrix} x_1 & x_2 \\ w_1 & w_2 \end{vmatrix} = x_1 w_2 - x_2 w_1$$

$$p_y z = \begin{vmatrix} y_1 & y_2 \\ z_1 & y_2 \end{vmatrix} = y_1 z_2 - y_2 z_1$$

$$p_y w = \begin{vmatrix} y_1 & y_2 \\ w_1 & w_2 \end{vmatrix} = y_1 w_2 - y_2 w_1$$

$$p_z w = \begin{vmatrix} z_1 & z_2 \\ w_1 & w_2 \end{vmatrix} = z_1 w_2 - z_2 w_1$$

Since $p_i i = 0$ and $p_i j = -p_j i$, only these six of the sixteen possible 2x2 determinants are non-zero and unique. These six numbers are called the Plucker coordinates of a line.

It can be shown that the determinant $|\mathbf{P1P2R1R2}|$ can be written as

$$|\mathbf{P1P2R1R2}| = p_x y r_z w + p_x z r_w y + p_x w r_y z + p_z w r_x y + p_w y r_x z + p_y z r_x w.$$

where the p's are the coordinates of the line **P1P2** and the r's are the coordinates of the line **R1R2**. (Note the change in sign by setting $p_w y = -p_y w$ and $r_w y = -r_y w$.) Using these line coordinates, each determinant can be calculated using only 6 multiplies and 5 adds!