CS 348C **Fall 1995-6**
Assignment 2 **Apostolos Lerios**

Due 11/16/95, 2:30p.

Notes

1. When you read this line, and provided you are taking the class for 3 units, please send mail to `tolis@cs.stanford.edu` and let me know that you have picked up the homework.

2. World Wide Web (WWW) documents referenced in this assignment can all be found under the course hierarchy

   `http://graphics.stanford.edu/courses/cs348c-95-fall/`

3. Please read the WWW page `requirements.html` for policies and regulations regarding your homework.

Problem 1: Polynomial Specification. *(8 points)*

We saw in class that a cubic polynomial $f(t)$ can always be specified unambiguously by four numbers f_1, f_2, m_1, and m_2 such that:

$$\begin{aligned} f(0) &= f_1 \\ f(1) &= f_2 \\ f'(0) &= m_1 \\ f'(1) &= m_2. \end{aligned}$$

That is, given four numbers f_1, f_2, m_1, and m_2, there is one and only one cubic polynomial $f(t)$ meeting all of the above conditions. In particular, this $f(t)$ is given by

$$f(t) = f_1 H_1(t) + m_1 H_2(t) + m_2 H_3(t) + f_2 H_4(t),$$

where $H_1(t), \ldots, H_4(t)$ are the Hermite basis polynomials for cubics.

Similarly, we may claim that we can always unambiguously specify the quadratic polynomial $f(t)$ by the following three numbers:

$$\begin{aligned} f(0) &= f_1 \\ f(1) &= f_2 \\ f'\left(\frac{1}{2}\right) &= m. \end{aligned}$$

Is this true? Follow these guidelines in stating your answer:

- If it is always possible to specify $f(t)$ unambiguously by f_1, f_2, and m, what are the corresponding basis functions? That is, find $Q_1(t)$, $Q_2(t)$, and $Q_3(t)$ such that

$$f(t) = f_1 Q_1(t) + m Q_2(t) + f_2 Q_3(t).$$

- If not, say so and prove your counter-claim. *Caution:* there are two parts to such a proof, namely showing that it not *always* possible to specify $f(t)$ by f_1, f_2, and m, and, whenever it is possible, we cannot specify $f(t)$ *unambiguously.*

Problem 2: Bernstein Basis. *(7 points)*

We saw in class that the canonical Bernstein polynomials for cubics are

$$\begin{aligned} B_1(t) &= (1-t)^3 \\ B_2(t) &= 3(1-t)^2 t \\ B_3(t) &= 3(1-t)t^2 \\ B_4(t) &= t^3. \end{aligned}$$

This means that if $\mathbf{b}_1, \ldots, \mathbf{b}_4$ are the canonical Bézier points of a cubic curve $\mathbf{p}(t)$, then

$$\mathbf{p}(t) = \mathbf{b}_1 B_1(t) + \mathbf{b}_2 B_2(t) + \mathbf{b}_3 B_3(t) + \mathbf{b}_4 B_4(t).$$

Now consider a cubic curve $\mathbf{q}(t)$ such that

$$\begin{aligned} \hat{\mathbf{q}}(r,r,r) &= \mathbf{b}_1 \\ \hat{\mathbf{q}}(r,r,s) &= \mathbf{b}_2 \\ \hat{\mathbf{q}}(r,s,s) &= \mathbf{b}_3 \\ \hat{\mathbf{q}}(s,s,s) &= \mathbf{b}_4, \end{aligned}$$

where $r \neq s$, and $\hat{\mathbf{q}}(t_1, t_2, t_3)$ is the polar form of $\mathbf{q}(t)$. In words, $\mathbf{b}_1, \ldots, \mathbf{b}_4$ are the Bézier points of the curve $\mathbf{q}(t)$ for the segment starting at $t = r$ and ending at $t = s$. Here, r and s may not be equal to 0 and 1, respectively; that is, we are considering (possibly) non-canonical Bézier points.

Derive the generalized Bernstein polynomials for $\mathbf{q}(t)$. That is, find $\tilde{B}_1(t), \ldots, \tilde{B}_4(t)$ such that

$$\mathbf{q}(t) = \mathbf{b}_1 \tilde{B}_1(t) + \mathbf{b}_2 \tilde{B}_2(t) + \mathbf{b}_3 \tilde{B}_3(t) + \mathbf{b}_4 \tilde{B}_4(t).$$

It is to be expected that each of these generalized Bernstein polynomials will depend on t, r, and s. Show your work.

Problem 3: Polar Forms. *(15 points)*

Consider a quadratic curve $\mathbf{p}(t)$ on the plane, defined by

$$\mathbf{p}(t) = (4t^2 + 6t + 7, 2t + 1).$$

Answer the following questions; show your work:

1. *(1 point)* What is the (2-)polar form $\hat{\mathbf{p}}(r, s)$ of $\mathbf{p}(t)$?

2. *(2 points)* Show that your answer to the previous part is a multiaffine, symmetric function, satisfying $\hat{\mathbf{p}}(t, t) = \mathbf{p}(t)$.

3. *(3 points)* Homogenize both the domain and the range of the polar form $\hat{\mathbf{p}}(r, s)$.

4. *(2 points)* Express $\hat{\mathbf{p}}[(1; 1+h), (s_w; s_x)]$ in terms of h, $\hat{\mathbf{p}}[(1; 1), (s_w; s_x)]$, and $\hat{\mathbf{p}}[(0; 1), (s_w; s_x)]$. Put more succinctly, express $\hat{\mathbf{p}}(1+h, \cdot)$ in terms of h, $\hat{\mathbf{p}}(1, \cdot)$ and $\hat{\mathbf{p}}(\delta, \cdot)$, where $\delta = (0; 1)$, and h is an arbitrary real number.

5. *(3 points)* Simplify
$$\hat{\mathbf{p}}(1+h, 1+h) - \hat{\mathbf{p}}(1, 1)$$
by using your answer to the previous part.

6. *(3 points)* Find $\mathbf{p}'(1)$ from first principles. That is,
   - write down the limit defining this derivative, using h as the limit variable.
   - express it in terms of polar forms,
   - simplify the numerator using your answer to the previous part,
   - calculate the limit as $h \to 0$, and
   - simplify the result by evaluating all of its terms.

7. *(1 point)* Verify your answer to the previous part by differentiating the univariate form of $\mathbf{p}(t)$, and evaluating the result at $t = 1$.

Problem 4: Parametric Continuity; Theory. *(15 points)*

Recall that when two curves $\mathbf{p}(t)$ and $\mathbf{q}(t)$ meet with parametric continuity C^n at $t = t_0$, this means that, for all integers $0 \le i \le n$,
$$\mathbf{p}^{(i)}(t_0) = \mathbf{q}^{(i)}(t_0)$$
where $\mathbf{p}^{(i)}(t_0)$ is the ith derivative of $\mathbf{p}(t)$ evaluated at $t = t_0$, and similarly for $\mathbf{q}^{(i)}(t_0)$. In words, parametric continuity requires that the two curves agree at $t = t_0$ it terms of position, velocity, acceleration, etc.

In terms of polar forms, the condition on continuity can be expressed as follows: two curves $\mathbf{p}(t)$ and $\mathbf{q}(t)$ of degree d join with C^n continuity at $t = t_0$ if and only if their d-polar forms agree on all sequences of polar arguments that include *at most* n arguments different from t_0.

A special case of this theorem concerns C^2 continuity of cubics. In particular, two cubic curves $\mathbf{p}(t)$ and $\mathbf{q}(t)$ join with C^2 continuity at $t = t_0$ if and only if
$$\hat{\mathbf{p}}(t_0, r, s) = \hat{\mathbf{q}}(t_0, r, s)$$
for all real numbers r and s.

1. *(10 points)* Prove the above special case of the general theorem. Recall that the ith derivative of a cubic curve $\mathbf{p}(t)$ is given in terms of its polar form by

$$\mathbf{p}^{(i)}(t) = 3 \cdot 2 \cdots (4-i)\hat{\mathbf{p}}(\underbrace{t, \ldots, t}_{3-i}, \underbrace{\delta, \ldots, \delta}_{i}).$$

   Also, recall that

$$\hat{\mathbf{p}}(t+h, r, s) = \hat{\mathbf{p}}(t, r, s) + h\hat{\mathbf{p}}(\delta, r, s)$$

   for all real numbers t, r, s, and h.

2. *(5 points)* The usual geometric construction for testing C^2 continuity between the cubic curves $\mathbf{p}(t)$ and $\mathbf{q}(t)$ at t_0 checks whether three equalities of the following form hold:

$$\begin{aligned}
\hat{\mathbf{p}}(t_0, t_0, t_0) &= \hat{\mathbf{q}}(t_0, t_0, t_0) \\
\hat{\mathbf{p}}(t_0, t_0, s_0) &= \hat{\mathbf{q}}(t_0, t_0, s_0) \\
\hat{\mathbf{p}}(t_0, r_0, s_0) &= \hat{\mathbf{q}}(t_0, r_0, s_0),
\end{aligned}$$

   where r_0 and s_0 are two fixed real numbers, neither equal to t_0. Prove that these three equalities alone are necessary and sufficient to guarantee C^2 continuity as defined by the above special case of the general theorem.

Caution: in both parts of this problem, you must show both *sufficiency* (the "if") and *necessity* (the "only if"); that is, your proofs should demonstrate equivalence, not mere implication.

Problem 5: Parametric Continuity; Geometry. *(15 points)*

In this problem, you are given plots showing the Bézier points of a cubic curve $\mathbf{p}(t)$; these points are drawn as black disks and are numbered 1 through 4 from the first to the last. You are also given the values of t that correspond to the first and last Bézier points. Similar information is provided for another cubic curve $\mathbf{q}(t)$; its Bézier points are drawn as hollow circles and labeled "a" through "d", in sequence.

You are also given a real number t_0; your task is to find the *highest* value of n for which $\mathbf{p}(t)$ and $\mathbf{q}(t)$ meet with continuity C^n at $t = t_0$. To this end, you should use polar interpolations and *graphically* test for continuity. You are expected to submit a clearly labeled drawing showing your work; tearing off and submitting the relevant pages of the assignment is acceptable. *Note*: the error tolerance in the figures is a quarter of an inch for most 300dpi laser printers.

Here are the five parts of this problem:

1. *(3 points)* The Bézier points of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in figure 1. $\mathbf{p}(0)$ and $\mathbf{p}(1)$ are respectively the first and last Bézier points of $\mathbf{p}(t)$; $\mathbf{q}(1)$ and $\mathbf{q}(2)$ are those of $\mathbf{q}(t)$. $t_0 = 1$.

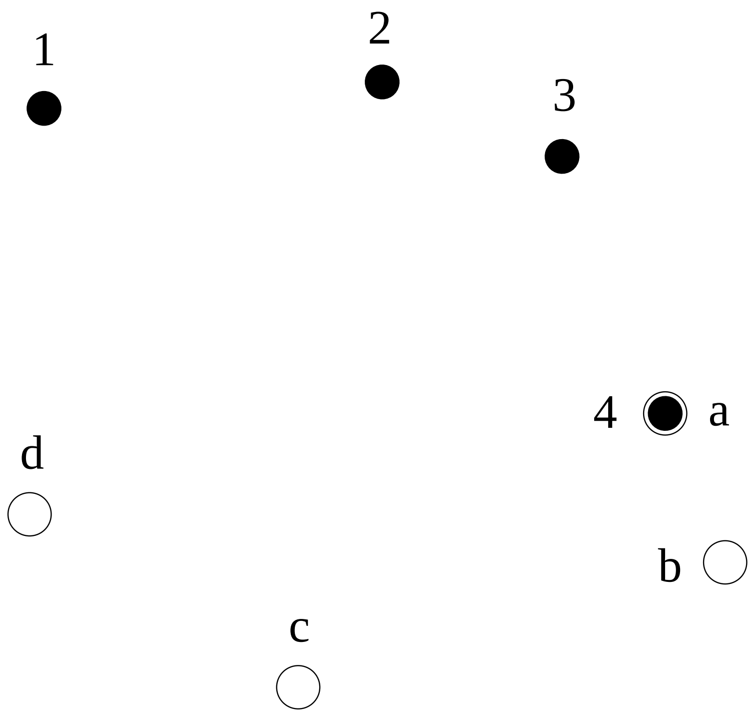

Figure 1: Bézier points for problem 5, parts 1 and 2.

2. *(3 points)* The Bézier points of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in figure 1. $\mathbf{p}(0)$ and $\mathbf{p}(2)$ are respectively the first and last Bézier points of $\mathbf{p}(t)$; $\mathbf{q}(2)$ and $\mathbf{q}(3)$ are those of $\mathbf{q}(t)$. $t_0 = 2$.

3. *(3 points)* The Bézier points of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in figure 2. $\mathbf{p}(0)$ and $\mathbf{p}(2)$ are respectively the first and last Bézier points of $\mathbf{p}(t)$; $\mathbf{q}(2)$ and $\mathbf{q}(3)$ are those of $\mathbf{q}(t)$. $t_0 = 2$.

4. *(1 point)* The Bézier points of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in figure 2. $\mathbf{p}(0)$ and $\mathbf{p}(2)$ are respectively the first and last Bézier points of $\mathbf{p}(t)$; $\mathbf{q}(2)$ and $\mathbf{q}(3)$ are those of $\mathbf{q}(t)$. $t_0 = 3$.

5. *(3 points)* The Bézier points of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in figure 3. $\mathbf{p}(0)$ and $\mathbf{p}(2)$ are respectively the first and last Bézier points of $\mathbf{p}(t)$; $\mathbf{q}(2)$ and $\mathbf{q}(3)$ are those of $\mathbf{q}(t)$. $t_0 = 2$.

6. *(2 points)* The Bézier points of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are shown in figure 3. $\mathbf{p}(0)$ and $\mathbf{p}(1)$ are respectively the first and last Bézier points of $\mathbf{p}(t)$; $\mathbf{q}(1)$ and $\mathbf{q}(2)$ are those of $\mathbf{q}(t)$. $t_0 = 2$.

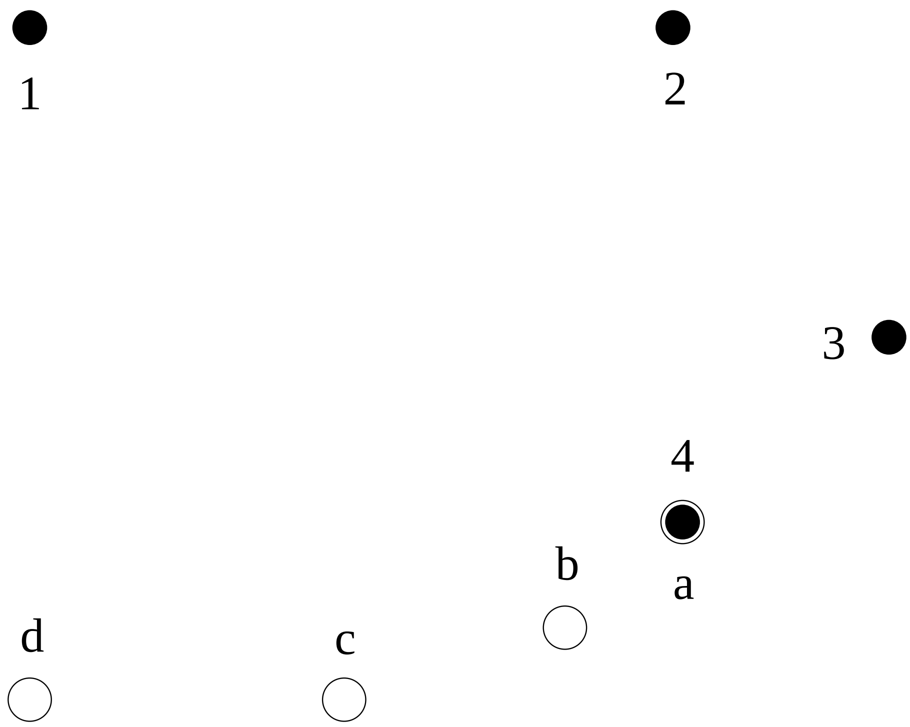

Figure 2: Bézier points for problem 5, parts 3 and 4.

Problem 6: B-Spline Curve Drawing. *(40 points)*

This is a programming problem, in part: you have to write the drawing portion of `scurvy`, a program that allows the user to visualize and manipulate B-spline curves in space. Also, you have to use `scurvy` to answer some simple questions.

Programming part. *(35 points)*

Here is what you have to do:

1. Log on to a Silicon Graphics machine which is equipped with the Open Inventor development environment.
2. Get the source code of `scurvy`, along with a parameter file, from the WWW. All the pertinent files are available as a `tar` (archive) file named

   `assignments/second/scurvy.tar`

   Save this file in some empty directory.
3. Unpack the archive by typing `tar xf scurvy.tar` in the command line. The following files should appear in your directory:

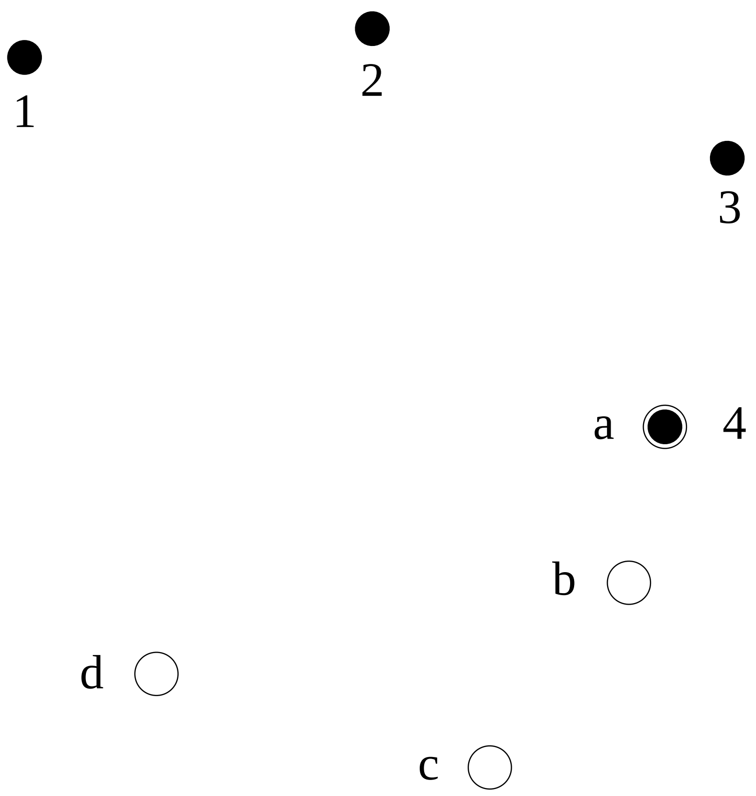

Figure 3: Bézier points for problem 5, parts 5 and 6.

- `Makefile`
- `draw.c`
- `scurvy.c++`
- `scurvy.h`
- `xsupport.c++`
- `xsupport.h`
- `question.sc`

You do not need your copy of `scurvy.tar` any longer.

4. Type `make`. Compilation and linking should succeed. You do not need the resulting executable named `scurvy`.

5. Peruse `scurvy.h`. It contains the headers and documentation of the functions you have to write.

6. Write the drawing code. To this end, modify the skeleton code of `draw.c`, mainly by writing the body of its five functions.

7. Make sure your code works correctly.

8. Comment your code adequately and carefully.

9. Mail your edited `draw.c` to `tolis@cs.stanford.edu`, entering `scurvy` in the subject line. If you discover any bugs after mailing in your code, just mail the updated copy; only your latest mail prior to the deadline will be used for grading purposes.

To help you in the development of your work, we provide a documented working executable of `scurvy`, along with sample parameter files; see the WWW page

`software/scurvy/index.html`

A copy of the executable and parameter files is also available on the Sweet Hall machines under

`/usr/class/cs348c/software/scurvy/`

This copy is also available via anonymous ftp to `ftp.stanford.edu` under

`/pub/class/cs348c/software/scurvy/`

Hints:

- Efficiency matters to the extent that the user interacts comfortably with the program. Do not spend hours devising your own, super-efficient fixed-point representation. Also, make sure that you activate the optimization flags in the `Makefile` when you are assessing efficiency.

- See the lecture notes from 10/31/95 for assigning de Boor points to spline segments. See the lecture notes from 10/17/95 for the de Boor algorithm, used to draw a spline segment given its de Boor points.

Written questions. ***(5 points)***

Peruse the provided parameter file `question.sc` and answer the following questions:

1. *(2 points)* What is the assignment of de Boor points to spline segments? To answer this question, construct an assignment table similar to the one which appears in the lectures notes from 10/31/95.

2. *(3 points)* What is the maximal continuity between segments $1\ldots2$ and $2\ldots3$ at $t=2$? By "maximal" we mean that your answer should be C^n, where n is as large as possible. Show your work. *Hint:* examine the spline parameters very carefully; this question is not as trivial as it may seem at first.