Topology Preserving Edge Contractions: What are they and how do we find them?

Jon McAlister

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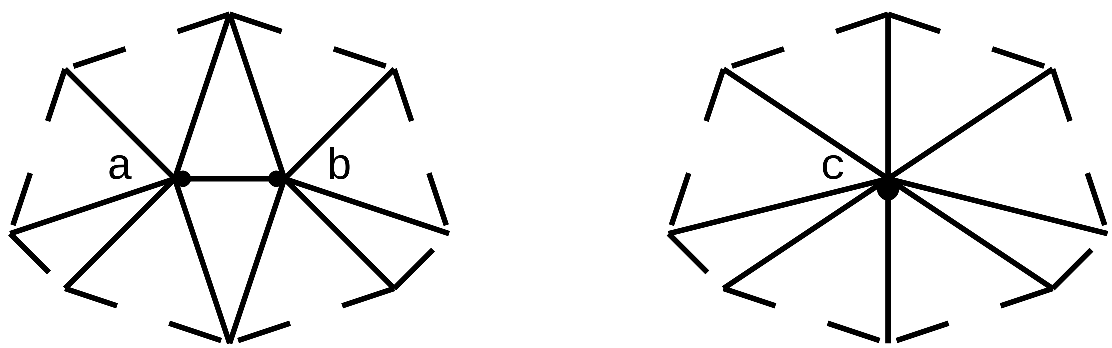
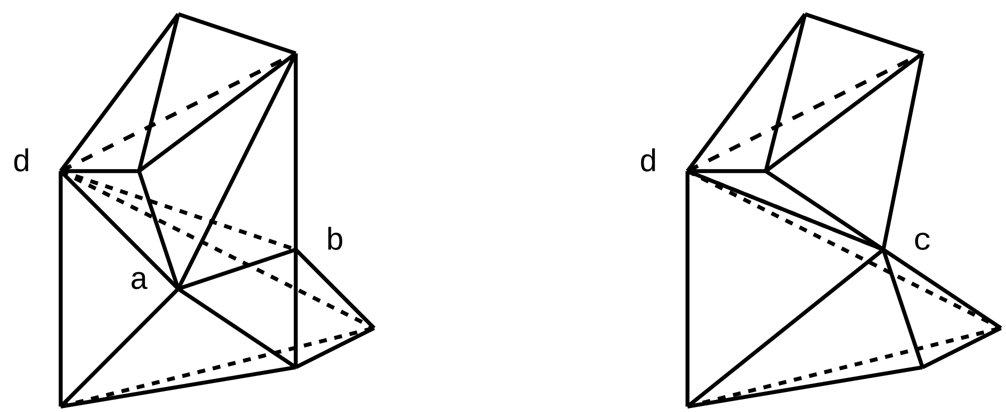
Edge contractions

- The contraction of an edge ab is a local transformation of K that replaces St $\overline{ab}$ = St $a \cup$ St b by the star of a new vertex, St c.



- What can go wrong?



Basic definitions

- The closure of $B \subseteq K$ is $\overline{B} = \{\tau \in K \mid \tau \leq \sigma \in B\}$.
- The star of $B \subseteq K$ is St $B = \{\tau \in K \mid \tau \geq \sigma \in B\}$.
- The link of $B \subseteq K$ is Lk $B = \overline{\text{St } B} -$ St $\overline{B}$.
- An edge is *contractable* if its contraction does not change the surface topology.
- A triangulation of a 2-manifold is *irreducible* if no edge is contractible.
- Two edge contractions in a 2-manifold are *independent* if they do not effect the same triangle. That is, St $\overline{ab} \cap$ St $\overline{cd} = \emptyset$.

Results overview

- *Topology Preserving Edge Contractions*, by Dey, Edelsbrunner, Guha, and Nekhayev, 1999.
  - Results in characterization of topology preserving edge contractions for simplicial complexes up to dimension 3 [1].
- *Hierarchy of Surface Models and Irreducible Triangulation*, by Cheng, Dey, and Poon, 2002.
  - Greedy algorithm to find $\Theta(n)$ independent topology preserving edge contractions in an orientable 2-manifold, each of which effect a small number of triangles [2].
  - Computing a topology preserving hierarchy of $O(n + g^2)$ size and $O(\log n + g)$ depth for an orientable 2-manifold [2].
  - Improved bound on the number of vertices in an irreducible triangulation of an orientable 2-manifold [2].

Simplicial map

- A *vertex map* for two complexes K and L is a map $f : \text{Vert } K \to \text{Vert } L$.
- The *barycentric coordinates* of a point $x \in \sigma$, $\sigma \in K$, are the unique reals $b_u(x), u \in \text{Vert } K$, so $b_u(x) \neq 0$ only if $u \leq \sigma$ and

$$\begin{aligned} x &= \sum_{u \in \text{Vert } K} b_u(x) \cdot u \\ 1 &= \sum_{u \in \text{Vert } K} b_u(x) \end{aligned}$$

- The *simplicial map* $\phi : |K| \to |L|$ for a vertex map f is defined by

$$\phi(x) = \sum_{u \in \text{Vert } K} b_u(x) \cdot f(u)$$

Unfoldings

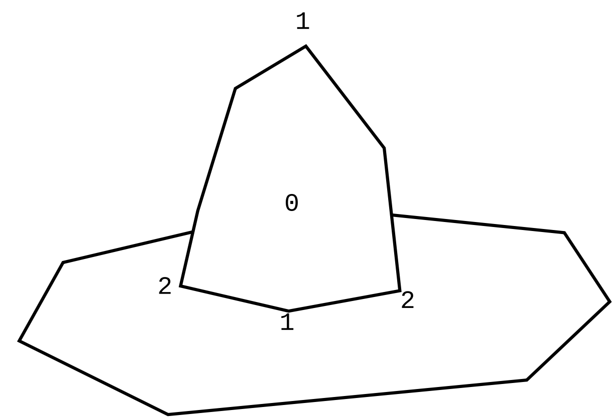
- $f : |K| \to |L|$ is a simplicial homeomorphism iff f is bijective and f^{-1} is also a vertex map.
- An edge contraction of ab can then be defined as a surjective simplicial map $\varphi_{ab} : |K| \to |L|$ defined by the surjective vertex map

$$f(u) = \begin{cases} u & \text{if } u \in \operatorname{Vert} K - \{a, b\} \\ c & \text{if } u \in \{a, b\} \end{cases}$$

- Note that outside $\left|\overline{\operatorname{St}}\ \overline{ab}\right|$, φ_{ab} is the identity, but inside it is not even injective.
- An *unfolding* of φ_{ab} is a simplicial homeomorphism $\psi : |K| \to |L|$.
- ψ is a *local unfolding* if it differs from φ_{ab} only inside $\left|\overline{\operatorname{St}}\ \overline{ab}\right|$.
- ψ is a *relaxed unfolding* if it differs from φ_{ab} only inside $\left|\overline{\operatorname{St}}\ \overline{\operatorname{St}}\ \overline{ab}\right|$.

Order and Boundary

- The *order* of σ is the smallest integer i for which there is a $(k-i)$ simplex η such that St σ and St η are combinatorially equivalent. Since the interior of η is homeomorphic to $\mathbb{R}^{k-i}$, the star of σ is homeomorphic to $\mathbb{R}^{k-i} \times \mathbb{X}$, for some topological space $\mathbb{X}$ of dimension i.



- The *j-th boundary* of a simplicial complex K is the set of simplices with order no less than j:

$$\text{Bd}_j K = \{\sigma \in K \mid \text{ord } \sigma \geq j\}$$

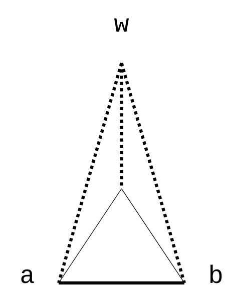
1-complexes

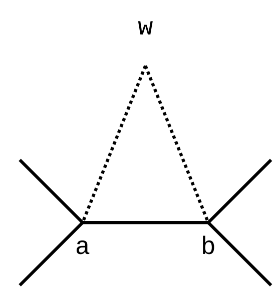
- For each i we extend the i-th boundary by adding a dummy vertex ω, and cones from ω to all simplices in the $(i+1)$-st boundary:

$$\mathrm{Bd}_i^\omega K = \mathrm{Bd}_i K \cup \omega \cdot \mathrm{Bd}_{i+1} K$$

- For a simplex $\sigma \in \mathrm{Bd}_i^\omega K$, we denote the link within $\mathrm{Bd}_i^\omega K$ as $\mathrm{Lk}_i^\omega \sigma$.
- For a 1-complex K, the following are equivalent:

1. (i) $\mathrm{Lk}_0^\omega a \cap \mathrm{Lk}_0^\omega b = \emptyset$.
2. (ii) φ_{ab} has a local unfolding.
3. (iii) φ_{ab} has an unfolding.





2-complexes

- For a 2-complex K then the following statements are equivalent:

  1. (i) $\mathrm{Lk}_0^\omega a \cap \mathrm{Lk}_0^\omega b = \mathrm{Lk}_0^\omega ab$, and
     $\mathrm{Lk}_1^\omega a \cap \mathrm{Lk}_1^\omega b = \emptyset$
  2. (ii) φ_{ab} has a local unfolding.

- They demonstrate a 2-complex which has neither a local nor a relaxed unfolding, but which does have an unfolding.

- For a 2-manifold the following statements are equivalent:

  1. (i) Lk a $\cap$ Lk b $=$ Lk ab.
  2. (ii) φ_{ab} has a local unfolding.
  3. (iii) φ_{ab} has an unfolding.

Steinitz' Theorem (1922)

- A *convex 3-polytope* is the convex hull of finitely many points in $\mathbb{R}^3$ that do not all lie in a common plane.
- The 1-skeleton is the subcomplex of all vertices and edges.
- A graph G is planar if it is isomorphic to a 1-complex in $\mathbb{R}^2$.
- A graph is 3-connected if the deletion of any two vertices together with their edges leaves the graph connected.
- *Steinitz' Theorem (1922)*: For every 3-connected planar graph there is a convex 3-polytope with an isomorphic 1-skeleton.

3-complexes

- For a 3-complex K then the following statements are equivalent:

  1. (i) $\mathrm{Lk}_0^\omega a \cap \mathrm{Lk}_0^\omega b = \mathrm{Lk}_0^\omega ab$,
     $\mathrm{Lk}_1^\omega a \cap \mathrm{Lk}_1^\omega b = \mathrm{Lk}_1^\omega ab$, and
     $\mathrm{Lk}_2^\omega a \cap \mathrm{Lk}_2^\omega b = \emptyset$
  2. (ii) φ_{ab} has a relaxed unfolding.

- For a 3-manifold the following statements are equivalent:

  1. (i) Lk a $\cap$ Lk b = Lk ab.
  2. (ii) φ_{ab} has a local unfolding.
  3. (iii) φ_{ab} has an unfolding.

Results overview

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References

[1] Tamal K. Dey, Herbert Edelsbrunner, Sumanta Guha, and Dmitry V. Nekhayev. Topology preserving edge contractions. Publ. Inst. Math (Beograd) (N.S.), 66 (1999), 23-45, 1999.

[2] Siu-Wing Cheng, Tamal K. Dey, and Sheung-Hung Poon. Hierarchy of surface models and irreducible triangulation. Available at http://cs468.stanford.edu, 2002.