Leopold Vietoris (1891–2002)

Heinrich Reitberger

On April 9, 2002, shortly before his 111th birthday, Leopold Vietoris died in a sanitarium at Innsbruck after a brief illness. The mathematical community has lost a well-known researcher. Vietoris was the recipient of several high awards.

Biography

L. Vietoris was born in Radkersburg (Styria) on June 4, 1891. After his graduation from the "Benediktinergymnasium" in Melk, he studied mathematics and descriptive geometry at the University and the Technical University in Vienna. In his sixth semester he heard a lecture on topology by W. Gross in 1912 based on the axioms of accumulation points by F. Riesz—extended by Gross. At the same time W. Rothe at the Technical University raised the question of the notion of manifold. Vietoris planned to create a geometrical notion of manifold with topological means. He was drafted in 1914 but continued working on his own on this problem. In September 1914 he was wounded, and after his recovery he was sent to the southern front. In 1916, while an army mountain guide, he obtained his first results, which he expanded during a three-month stay in Vienna (spring semester 1918), where he read for the first time Hausdorff's classic (published in 1914). On November 4, 1918, he became an Italian prisoner of war. Due to decent treatment, he was able to complete his thesis, which after his release he submitted to G. v. Escherich and W. Wirtinger in December 1919 at the University of Vienna. Before that, he had passed the exam for high-school teachers. During the following period of teaching he received a postcard from Escherich congratulating him on his thesis and offering him an assistant position at the Technical University in Graz. Two years later Vietoris received his Habilitation in Vienna on the recommendation of H. Hahn.

In 1925 Vietoris started working in combinatorial topology. He spent three semesters as a Rockefeller fellow with L. E. J. Brouwer in Amsterdam, where P. Alexandrov and K. Menger (whom he knew as a student from Vienna) were staying at the same time. It was during this stay that he began writing the papers on algebraic topology for which he is best known (Mayer-Vietoris sequence and so forth). In 1927 he followed a call to Innsbruck as associate professor, in 1928 he went back to the Technical University in Vienna as full professor, and in 1930 he finally settled in Innsbruck.

In the autumn of 1928 Leopold Vietoris married Klara Riccabona. She later died after giving birth to their sixth daughter. In 1936 he married Maria Riccabona, Klara's sister, who thenceforth was a mother to her nieces and a devoted spouse. She died shortly before her husband.

Foundations of General Topology[1]

To avoid Hausdorff's countability axioms, Vietoris added to the neighborhood axioms his separation axiom of "regularity"; defined filter base ("Kranz" = wreath), directed set ("orientierte Menge"), nets, and the related convergence concept; and introduced the modern notion of compactness (under the name "lückenlos" = without gaps).

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[1] *See also [Rei97], [Rei02].*

Discussing the notion of directed set in 1937, G. Birkhoff [Bir37] wrote the condition "(D3) given $\alpha \in A$, $\beta \in A$, there exists $\gamma \in A$ satisfying $\gamma > \alpha$ and $\gamma > \beta$" and remarked, "It is primarily condition (D3) which was due to Moore and Smith, and which distinguishes directed sets from other 'partially ordered sets'". Indeed, (D3) occurred as the "composition property" in a paper of E. H. Moore and H. L. Smith [MS22][2], but priority belongs to Vietoris, who introduced this concept as "oriented set" in [Vie21, p. 184]. Interestingly enough, Birkhoff quoted this paper, concerning the separation axioms on p. 174, but seems not to have read any further!

Whereas Moore and Smith considered only generalized sequences with numerical values, Vietoris studied right from the start "sets of second order", i.e., systems of sets under Zermelo's axioms, indexed by a directed set, and gave the definition: *A set of second order is called a "Kranz" [wreath], if the intersection of any two elements contains again an element $\neq \emptyset$.* With respect to inclusion, a "Kranz" forms a directed system of sets, and vice versa, the remainders $R(B) := \{x \in M : b < x \ \forall b \in B\} \neq \emptyset$ of a directed set form a "Kranz". So Vietoris developed in parallel today's theory of convergence for generalized set sequences (nets) and filter bases through comparison with the directed set of neighborhoods.

Although Birkhoff reinvented the notion of filter base in 1935 [Bir35], it is H. Cartan who has generally been thought of as the creator of the concept on the basis of his 1937 papers [Car37a, Car37b].

Vietoris was thus the father of the modern convergence concepts (and more (see below)), yet his name is not mentioned in the "Historical Notes" in Bourbaki's *General Topology*. One explanation may be that in the—otherwise excellent—encyclopedia article "Relations between the different branches of topology", published in 1930 by Tietze and Vietoris [Vie31], the notions of "Kranz" and "orientierte Menge" are missing!

H. Cartan wrote in his second note on filters [Car37b] that "Chevalley and Weil have led me to remark that the definition of a compact space by the property of Borel-Lebesgue is equivalent to the following: *E is compact if every filter on E has at least one cluster point.*" In [Vie21], Vietoris had defined this modern notion of compactness—but under the redundant general assumption of regularity, as Urysohn remarked—under the name "lückenlose Menge" (set without gaps) through an analogous property of nets and had given the characterization by filter bases: *A wreath without last element always has a proper cluster set.* Vietoris also gave other equivalent formulations but not the one by the covering property. This general notion of compactness was then called "bicompact" by Alexandrov and Urysohn from 1923 onwards.

Of the theorems that Vietoris proved for compact spaces in [Vie21], he himself considered "Satz (27)" to be his most important result: *Two closed disjoint sets A and B have two enclosing sets[3] that have no point in common* (in modern terminology: a compact space is normal). It is remarkable that this separation property (later Tietze's normality) was mentioned here for the first time and proved for compact sets! For the proof, Vietoris showed that the sets U_α enclosing A form a filter base, as do the sets V_β enclosing B, and the sets $U_\alpha \cap V_\beta$ if nonempty. To reach a contradiction, he used the regularity; compare the verbatim same proof in Bourbaki's *General Topology*, 9.2, Prop. 2.

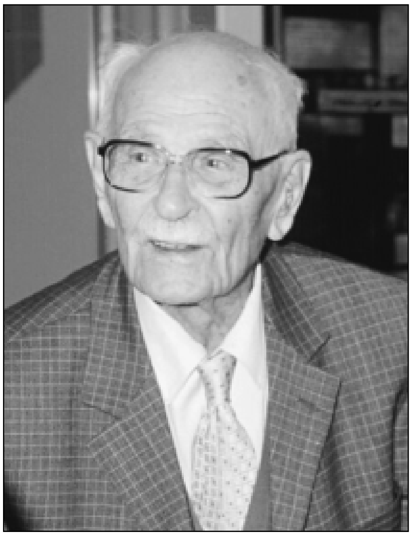
Vietoris started his work [Vie21] with the neighborhood axioms (including Hausdorff's separation property) and added Axiom (E): *A neighborhood U_x of a point x always contains a neighborhood W_x of x such that each point of the complement of U_x, including one of its neighborhoods, lies in the complement of W_x.* In a footnote he indicated that Axiom (E) is not mentioned by Hausdorff, who had instead two countability axioms. Today's somewhat different definition of regularity is due to

Ph.D. Students of Leopold Vietoris

Kurt Hellmich,
Funktionen, deren Werte Mengen sind (1939)
Martha Petschacher,
Tafeln hypergeometrischer Funktionen (1946)
Hiltrud Jochum,
Die Cayleyschen Formeln in der Kreisgeometrie und die Brennpunkte in der Gaußschen Ebene (1952)
Johann Leicht,
Zur intuitionistischen Algebra und Zahlentheorie (1952)
Walter Dürk,
Der Strukturkomplex $t_1 t_2 = b^2$ (1953)
Helmut Grömer,
Über den Begriff der Wahrscheinlichkeit (1954)
Eva Ambach,
Der Größenfehler einer durch das Adamssche Interpolationsverfahren gewonnenen Näherungslösung einer Differentialgleichung (1957)
Gerhard Riege,
Das Axiom von Pasch in konvexen Räumen (1957)
Fortunat Pescolderung,
Über eine Fehlerabschätzung zur numerischen Integration gewöhnlicher Differentialgleichungen (1958)
Egon Steuer,
Zur Kreisgeometrie ebener algebraischer Kurven (1960)

[2] *Previous works of E. H. Moore already contain the suggestion of a general convergence theory but not condition (D3)!*

[3] *An enclosing set of a set A contains a neighborhood of each point of A.*



Photograph courtesy of Heinrich Reitberger.

Leopold Vietoris

Tietze in 1923. As Urysohn remarked, the term "regularity" goes back to Alexandrov.[4]

The attempts to create a convenient notion of manifold led Vietoris to look for a spatial structure on the power set of a topological space. In "Regions of second order" [Vie22] he defined on the collection of all nonempty closed subsets $\mathbf{CL}(S)$ of a topological space $(S, \mathcal{T})$ a topology as follows (compare Nadler [Nad78]). For each finite collection $U_1, \ldots, U_n \in \mathcal{T}$, let $\langle U_1, \ldots, U_n \rangle$ denote the set of all A in $\mathbf{CL}(S)$ such that $A \subset \cup_{i=1}^{n} U_i$ and $A \cap U_i \neq \emptyset$ for each $i = 1, \ldots, n$; the sets $\langle U_1, \ldots, U_n \rangle$ form the base of a topology on $\mathbf{CL}(S)$. In the case of a compact connected metric space X this "Vietoris topology"on $\mathbf{CL}(X)$ coincides with topology induced by the so-called "Hausdorff metric". Recently, this metric has again become important, for example in fractal image compression—so to speak, a jump from "hyperspace to cyberspace"!

Algebraic Topology

Algebraic topology develops methods for deciding with algebraic tools whether two topological spaces are homeomorphic. The applications range from simple-sounding questions, such as whether a product like the one in the complex numbers exists also in higher dimensions, to the theory of knots and its use in particle physics and biochemistry. Since Poincaré's time topologists have tried to find appropriate invariants—first for simplicial complexes, then more generally for metric spaces, as Vietoris showed us [Vie27][5], and finally for general spaces by means of coverings.

Now we come to the so-called *Vietoris complex*. Let X be a metric space. An (ordered) n-dimensional ϵ-simplex σ^n of X is an $(n+1)$-tuple of points $e_0, e_1, \ldots, e_n$ in X such that the distance of any two is less than ϵ. Let G be an abelian group. A formal linear combination $\sum_i g_i \sigma_i^n$ of ϵ-simplices with coefficients $g_i \in G$ is called an ϵ-chain in X. The boundary of an ϵ-simplex $\sigma^n = [e_0, \ldots, e_n]$ is defined by

$$\partial \sigma^n := \sum_i (-1)^i [e_0, \ldots, \hat{e}_i, \ldots, e_n].$$

This is again an ϵ-chain. The boundary of any ϵ-chain is defined by linear extension. The ϵ-chains with zero boundary are called ϵ-cycles. An ϵ-chain x^n is called η-homologous to zero in X, written $x^n \sim_\eta 0$, if $x^n = \partial y^{n+1}$ for an η-chain y^{n+1} in X. Vietoris called a sequence $z^n = (z_1^n, \ldots, z_k^n, \ldots)$ of ϵ_k-cycles z_k^n in X *fundamental* if $\epsilon_k \to 0$ (for $k \to \infty$) and for all $\epsilon > 0$ there exists $N(\epsilon)$ such that for all $l, m > N(\epsilon)$, we have $z_l^n \sim_\epsilon z_m^n$, that is, $z_l^n - z_m^n \sim_\epsilon 0$ in X. The fundamental sequences form a group $Z_n(X, G)$. A fundamental sequence z^n is called homologous to zero if for all $\epsilon > 0$ there exists an N such that $z_k^n \sim_\epsilon 0$ for all $k \geq N$.

The quotient group

$$H_n(X, G) = Z_n(X, G)/\{\text{null sequences}\},$$

the *n-th homology group*, was the central object for the further studies of Vietoris[6] (cf. Hirzebruch [Hir99] and Mac Lane [ML86]).

First an anecdote: Until World War Two, the Vietoris complex and V(ietoris)-cycles were part of the standard knowledge of all topologists (cf. [Lef42]). Twenty years ago the complex was reinvented by E. Rips while studying hyperbolic metric groups, and M. Gromov used it in his fundamental works on these groups. Finally in 1995, J.-Cl. Hausmann saw that this concept goes back to Vietoris, and so he called it the *Vietoris-Rips complex*.

An important tool to determine the homology groups of a space is a process to compute them from simpler pieces. This is given by the *Mayer-Vietoris sequence*—the best-known result connected with the name Vietoris.

Theorem (Mayer-Vietoris sequence). *Let K_1 and K_2 be subcomplexes of a simplicial complex K. Then the sequence of homology groups*

$$\begin{aligned} \ldots \to H_q(K_1 \cap K_2) \to H_q(K_1) \oplus H_q(K_2) \\ \to H_q(K_1 \cup K_2) \to H_{q-1}(K_1 \cap K_2) \to \cdots \end{aligned}$$

is an exact sequence.

Concerning the genesis of this theorem, we may let the principals speak for themselves. Mayer [May29]: "I was introduced to topology by my colleague Vietoris, whose lectures in 1926–7 I attended at the local university. In many talks about this field Vietoris gave me a lot of hints for which I am very grateful." Vietoris [Vie30]: "W. Mayer, whom I told about the problem as well as the conjectured result and a way to its solution, has solved the question, as far as it concerns Betti numbers, in a somewhat different way in these Monatshefte. In what follows, I will return to my original idea and use it for the solution in the general case." Thus

[4] *For R^2 the regularity property results already from an axiom of R. L. Moore, as mentioned by Chittenden.*

[5] *A short version appeared in* Proc. Amsterdam ***29*** *(1986), 1008-13.*

[6] *Vietoris commented that these studies were inspired by an oral remark of Brouwer.*

Vietoris calculated the homology *groups* and not just the Betti numbers (i.e., the ranks of the groups).

In May 1946 J. Leray introduced today's central notions of sheaf, sheaf cohomology, and spectral sequence. His motivation was the following situation [Die89]: Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The main problem is (after Leray) to relate the homology of X and the homology of Y, perhaps under some restrictions on f. Apparently Leray was unaware in 1946 that Vietoris had obtained such a result twenty years earlier together with the definition of homology groups for the case of compact metric spaces [Vie27].

Indeed, Leray first turned his attention to topology in earnest during World War Two, when access to the literature was problematic. The story, as related in [BHL00], is that Leray was imprisoned during the war as a French officer in a camp in the Waldviertel (Lower Austria) as head of the "prisoner university" Edelbach-Allentsteig. To avoid being forced to work for the German army, Leray remained silent about his knowledge of fluid dynamics and instead represented himself as a topologist.

We now come to the second part of Vietoris's paper: the *mapping theorem* (in the formulation of S. Smale, who generalized it in 1957).

Theorem (Vietoris-Begle). *Let X, Y be compact metric spaces, $f : X \to Y$ surjective and continuous. Suppose that for all $0 \le r \le n-1$ and all $y \in Y$ the reduced homology groups $\tilde{H}_r(f^{-1}(y))$ vanish (in [Vie27] Vietoris used the coefficient group $G := \mathbf{Z}/2\mathbf{Z}$). Then the induced homomorphism*

$$f_\star : \tilde{H}_r(X) \to \tilde{H}_r(Y)$$

is an isomorphism for $r \le n-1$ and an epimorphism for $r = n$.

The fibers are assumed to be *acyclic* ("without holes"), in other words, they have the same homology as a single point. For further explanation we consider the situation in topological vector spaces: For a nonempty subset A we have

$$\text{convex} \Longrightarrow \text{starshaped} \Longrightarrow \text{contractible}$$
$$\Longrightarrow \mathit{acyclic} \Longrightarrow \text{connected}.$$

In 1950, E. G. Begle extended the theorem to compact Hausdorff spaces. For the historical developments, especially the application of the mapping theorem to derive fixed-point theorems for correspondences, see [Rei01].

Functional and Differential Equations

In his paper [Vie44], Vietoris reduced the functional equations for the trigonometric functions to the equation

(1) $A(x + \xi) = A(x)A(\xi)$

for a complex function $A(x) = \exp\{u(x) + i\varphi(x)\}$, where u and φ are real functions of a real variable x, satisfying

(2) $u(x + \xi) = u(x) + u(\xi)$,

(3) $\varphi(x + \xi) \equiv \varphi(x) + \varphi(\xi) \pmod{2\pi\mathbf{Z}}$.

By using a Hamel basis, he found a new solution to (3), simpler than an earlier one by van der Corput. In 1957 Vietoris used the Cauchy functional equation (3) to give a remarkable proof—as Aczél says—of the limit $\lim_{x\to 0}(\sin x)/x = 1$.

In a series of papers, Vietoris treated the solution of ordinary differential equations by mechanical means, beginning with a modification of the Picard method of successive iterations.

Probability

The object of Vietoris's papers on probability theory was the introduction of nine axioms governing the "eher"-relation $\le$, where $a_A \le b_B$ corresponds to the intuitive idea "outcome a in trial A is not more probable than outcome b in trial B", and the derivation of the laws of classical probability from these. His approach to probability is the same as B. O. Koopman's. Alluding to the problem of obtaining original sources in wartime, Vietoris pointed out in a footnote that he had not seen Koopman's paper and only learned about it later from a review.

Positive Trigonometric Sums

Vietoris proved important inequalities in three papers *Über das Vorzeichen gewisser trigonometrischer Summen* [Vie58, Vie59, Vie94], the last having been written at the youthful age of 103 years!

Theorem. *Let $a_0, a_1, \dots, a_n$ and t be real numbers. If*

(1) $a_0 \ge a_1 \ge \cdots \ge a_n > 0$ and

(2) $a_{2k} \le \frac{2k-1}{2k} a_{2k-1}$ $(1 \le k \le \frac{n}{2})$, then

(3) $$\sum_{k=1}^{n} a_k \sin kt > 0 \quad \text{and}$$
$$\sum_{k=0}^{n} a_k \cos kt > 0 \quad (0 < t < \pi).$$

Putting $a_0 = 1, a_k = \frac{1}{k}$ $(k = 1, \dots, n)$ gives the Fejér-Jackson inequality $\sum_{k=1}^{n} \frac{1}{k} \sin kt > 0$ $(0 < t < \pi)$ and the W. H. Young inequality $1 + \sum_{k=1}^{n} \frac{1}{k} \cos kt > 0$ $(0 < t < \pi)$.

R. Askey reports in [Ask98] his surprise in seeing (3) for the first time and in learning that the Fejér inequality is not sharp. Then he discusses some problems suggested by Vietoris's inequalities and shows how one of them leads to the derivation of the hypergeometric summation formula and to other summation formulas.

Applications

By conferring honorary doctor degrees of technical sciences, the Technical University of Vienna in 1984 and the Technical Faculty of the Innsbruck University in 1994 acknowledged the contributions of Vietoris to practical applications. These concern his works on orientation in mountainous terrain by differential geometric means, the strength of the alpine ski, and the physics of block glaciers.

Final Remarks

Leopold Vietoris's fundamental contributions to general as well as algebraic topology, and also to other branches of the mathematical sciences, have made him immortal in the world of science. As a person, he was outstandingly humble and grateful for his well-being, which he also wished and granted his fellow humans. He devoted his spare time to his large family, religious meditation, music, and his beloved mountains. On the other hand, administrative duties were not Vietoris's favorite tasks, as he pointed out in a letter to L. E. J. Brouwer in 1947: "As dean I am overwhelmed with administrative matters to such an extent that I often have to hold my lectures inadequately prepared and don't have any time for scientific research. Luckily, the term will soon be over and then I hope to be a scientist again and not a bureaucrat." In research Vietoris was a "lone fighter": Only one of his more than seventy mathematical papers has a coauthor. Half of the papers were written after his sixtieth birthday.

A long life has fulfilled itself. Beside the grief comes our thankfulness!

Acknowledgements

I thank Ottmar Loos, who, following G. Lochs, currently occupies Vietoris' professorial chair, and the editors for their help with the English version of this obituary and many other helpful suggestions. A German obituary that includes a complete list of Vietoris's publications will appear in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* **104** (2002).

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