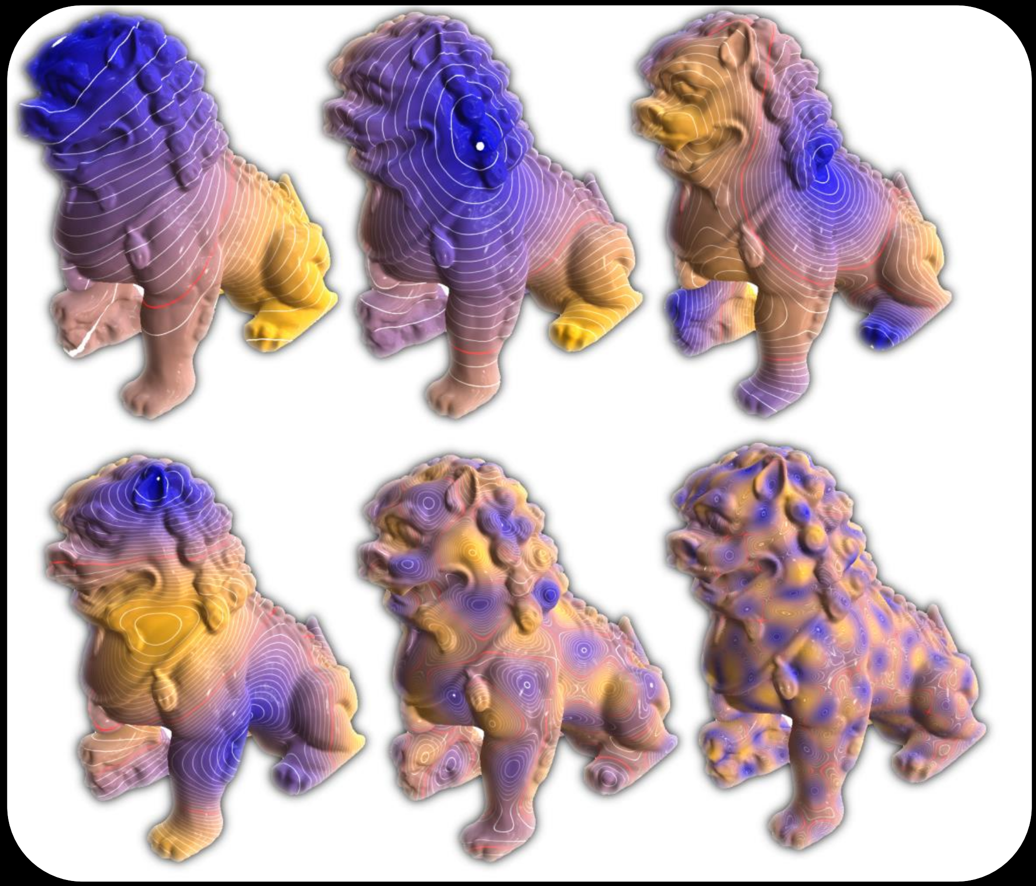
http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png

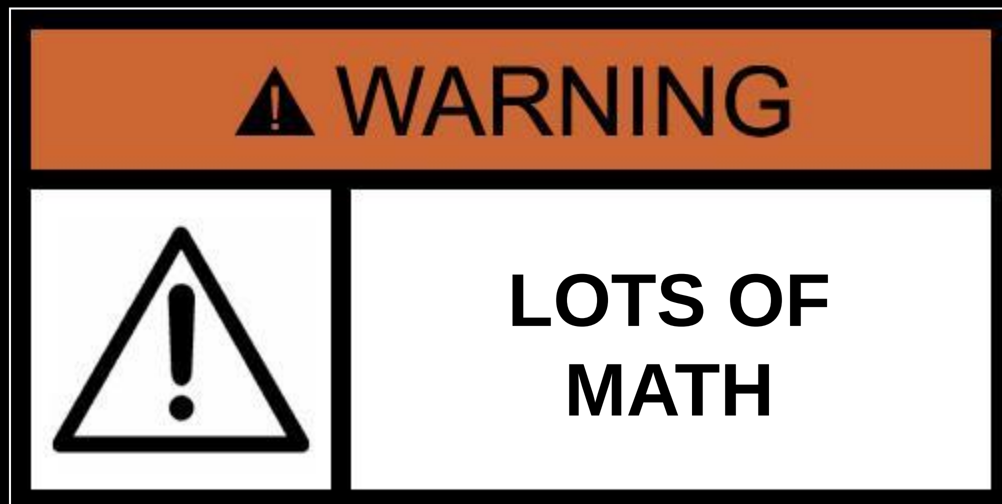


Discrete Laplacians

CS 468, Spring 2013
Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher


WARNING
SIGN
MISTAKES
LIKELY


WARNING
LOTS OF
MATH

Today's Focus

$$\Delta$$

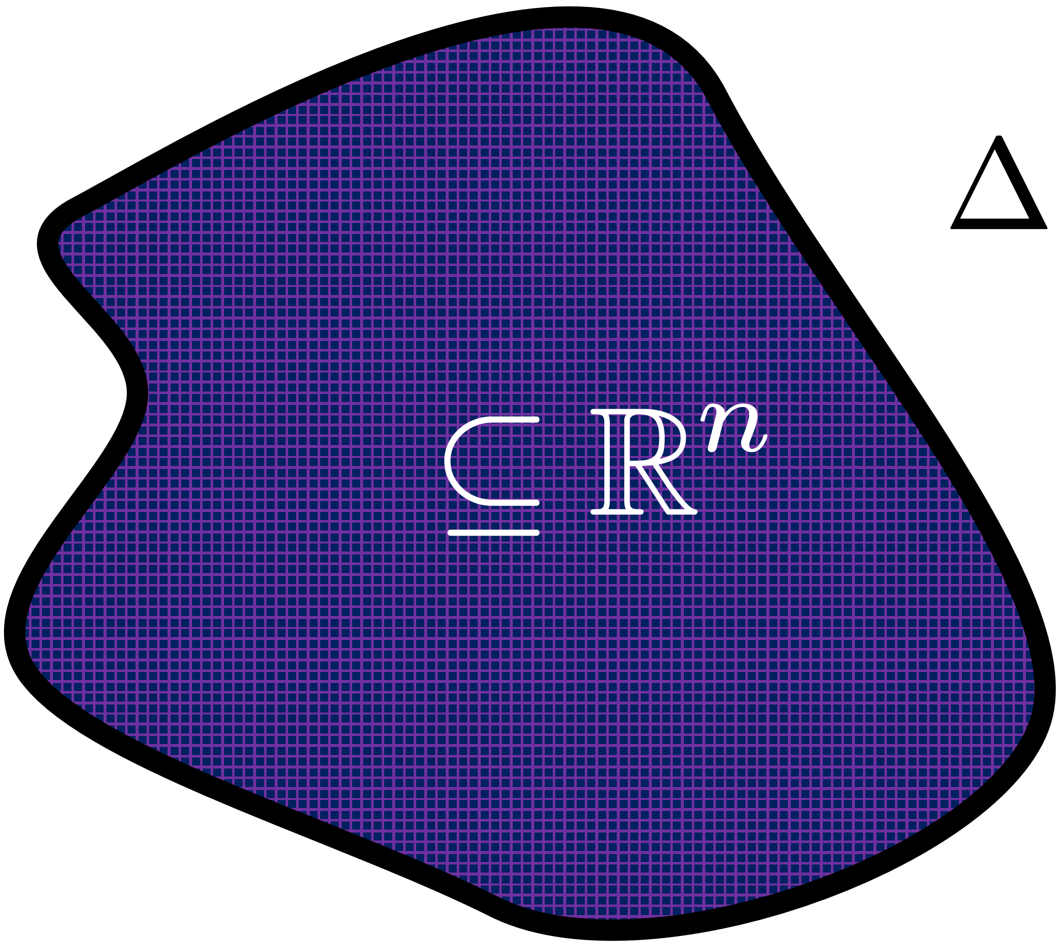
The Laplacian

Linear Functional on $C^{\infty}(M)$

$$f \in C^{\infty}(M) \longrightarrow \Delta \longrightarrow \Delta f \in C^{\infty}(M)$$

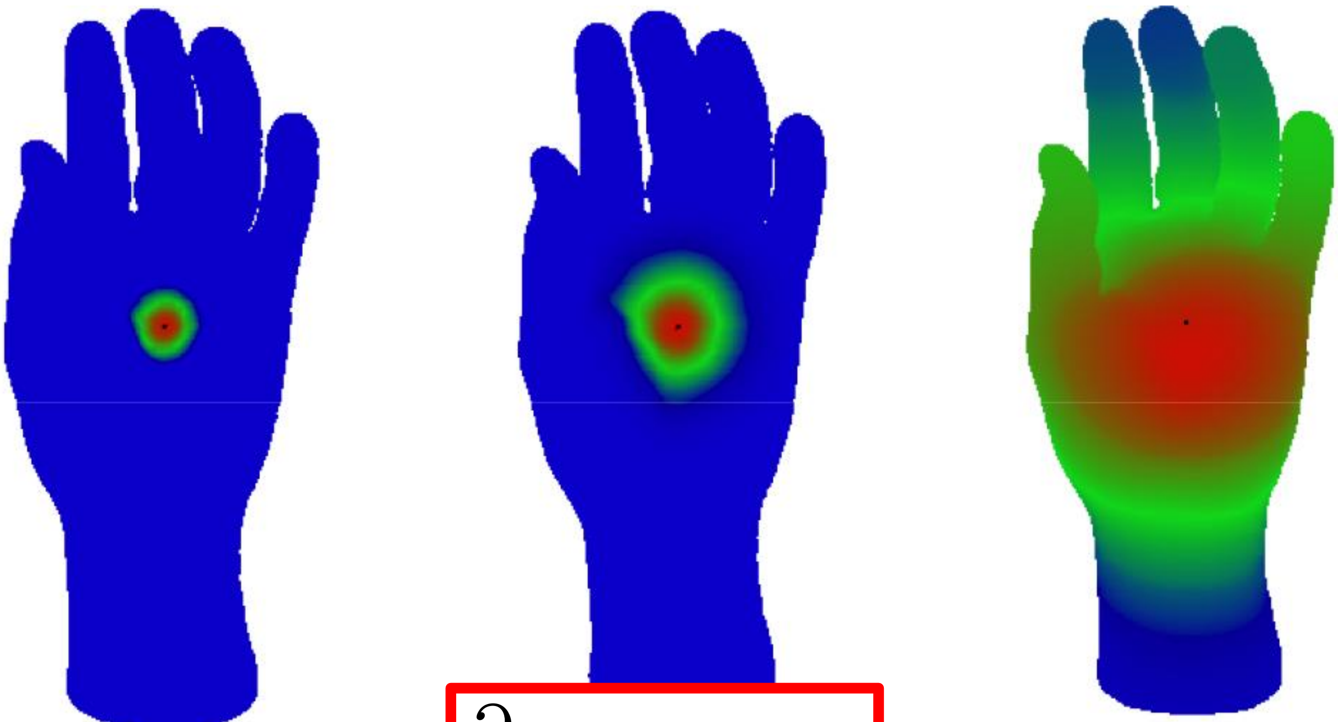
The Laplacian

Laplacian on R^n



$$\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}$$

Connection to Physics

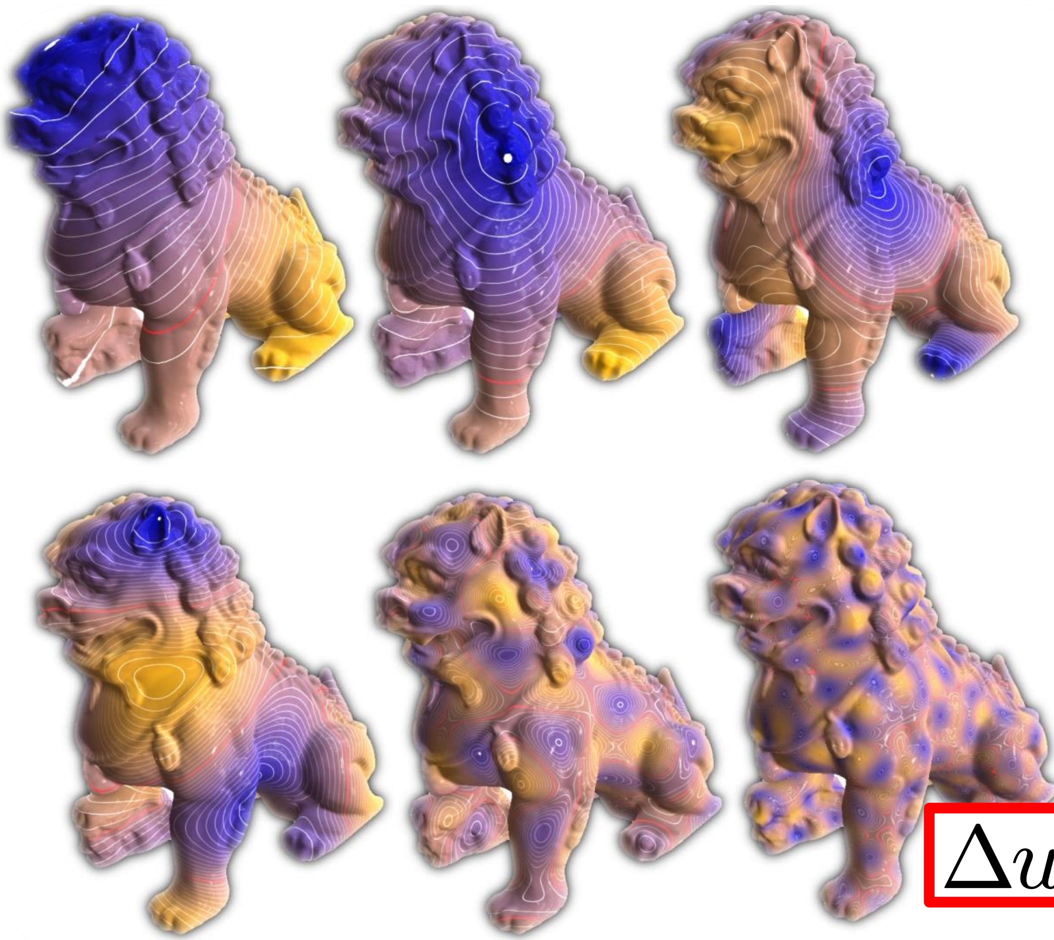


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Connection to Physics



$$\Delta u = \lambda u$$

Vibration modes

Defining the Laplacian

$$\text{“}\Delta f = \text{div grad } f\text{”}$$

Defining the Laplacian

$$\text{“}\Delta f = \text{div grad } f\text{”}$$

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

?!

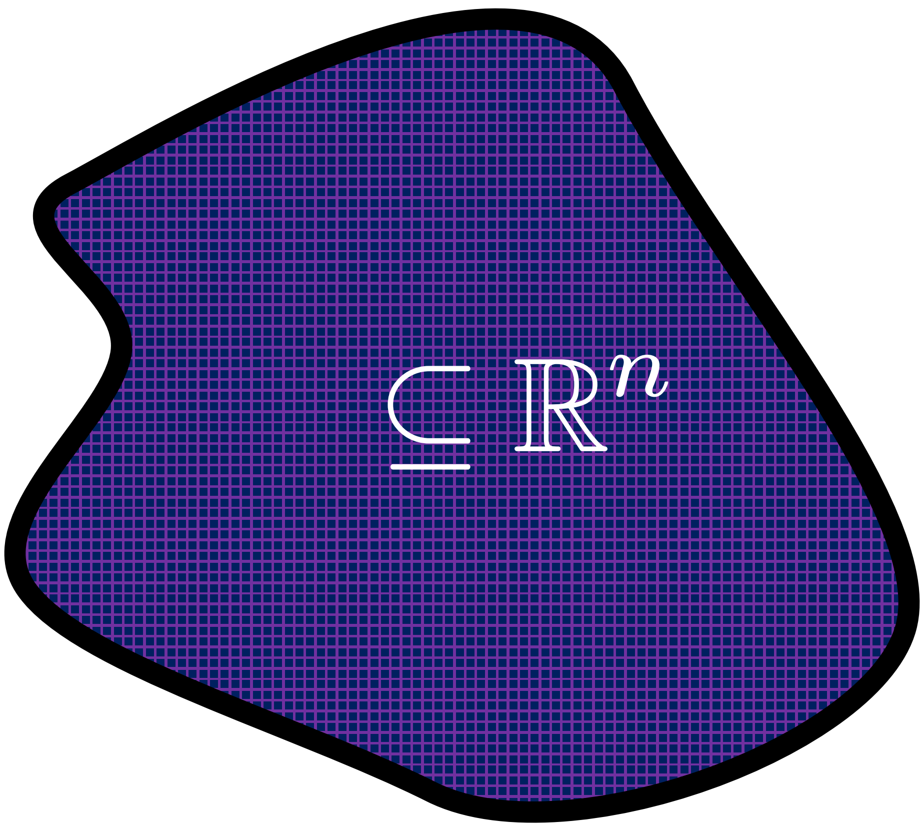
Defer: Divergence

Cleaner notation in a few lectures.

"Is lecture over?!"

Integration by Parts to the Rescue

$$\int_{-} f\Delta g \; dA = \text{boundary terms} - \int_{-} \nabla f \cdot \nabla g \; dA$$



Problem

Laplacian is a *differential* operator!

L^2 Dual of a Function

$$f : M \to \mathbb{R}$$

$$\Downarrow$$

$$\mathcal{L}_f : L^2(M) \to \mathbb{R}$$

$$\mathcal{L}_f[g] = \int_M fg \, dA$$

"Test function"

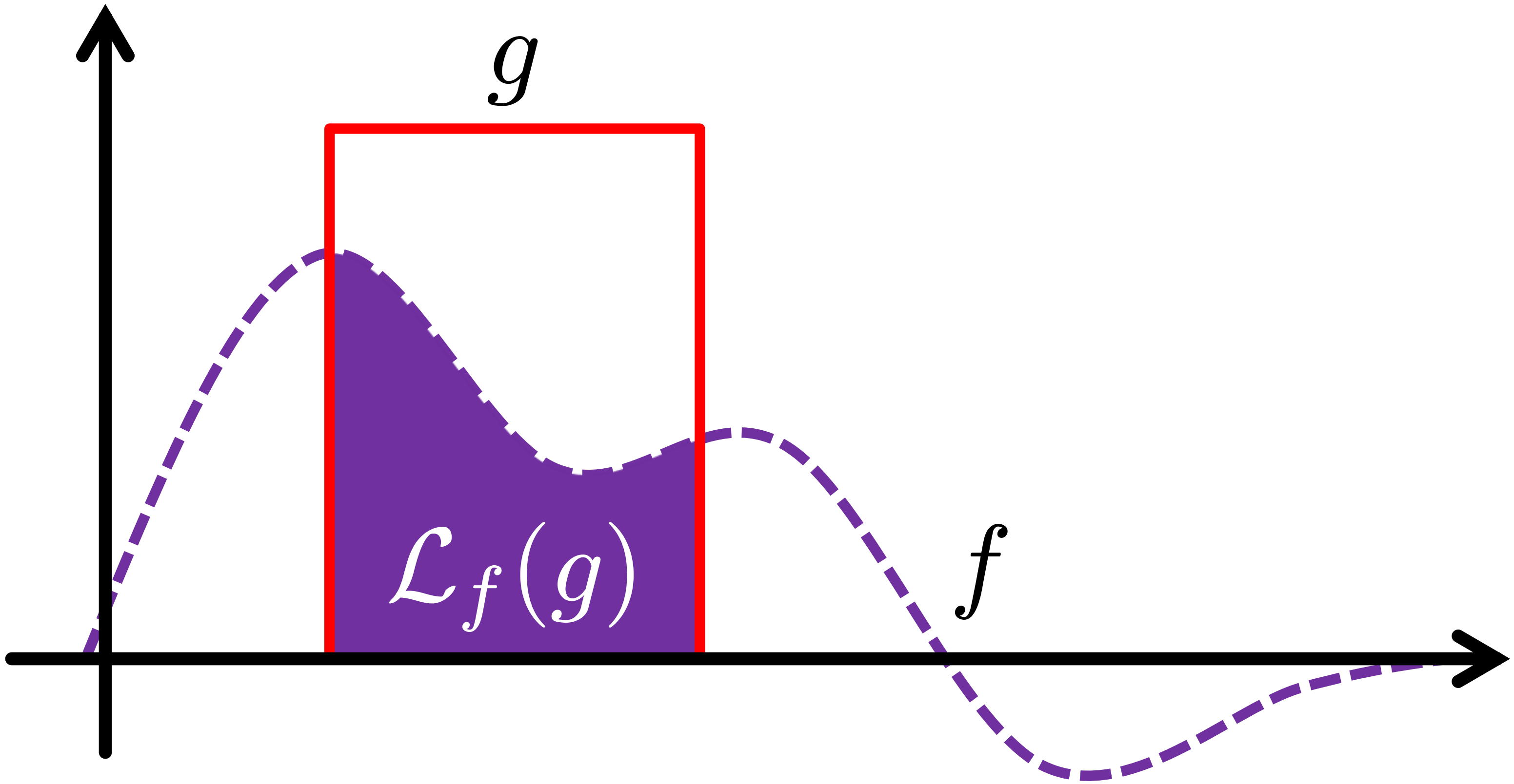
Set of Test Functions

$$\{g \in L^\infty(M) : g|_{\partial M} \equiv 0\}$$

Often $\partial M = \emptyset$

Dirichlet boundary conditions

Observation



Can recover function from dual

Dual of Laplacian

$$\{g \in L^\infty(M) : g|_{\partial M} \equiv 0\}$$

$$\begin{aligned}\mathcal{L}_{\Delta f}[g] &= \int_M g \Delta f \; dA \\ &= -\int_M \nabla g \cdot \nabla f \; dA\end{aligned}$$

Use Laplacian without evaluating it!

Galerkin's Approach

Choose one of each:

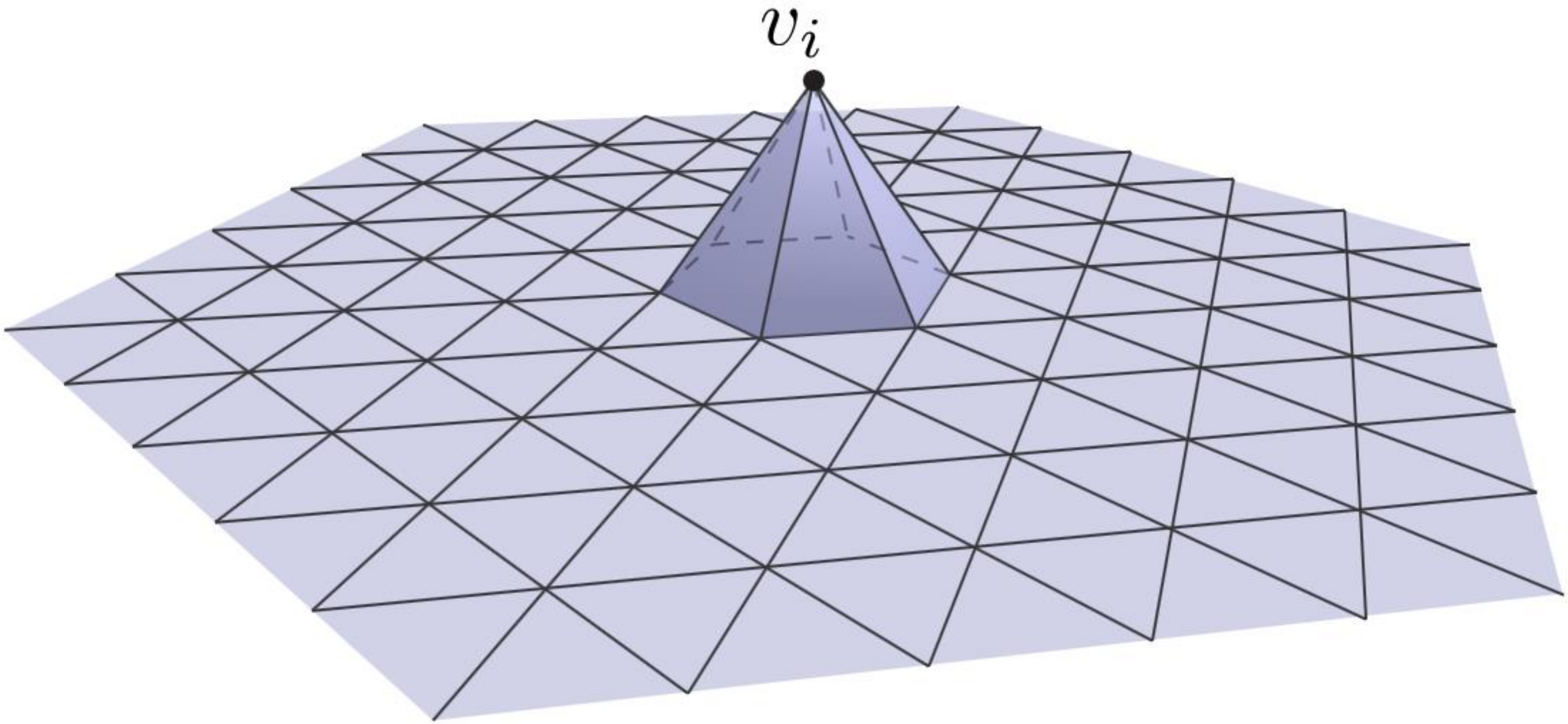
- Function space
- Test functions

Often the same!

One Derivative is Enough

$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \; dA$$

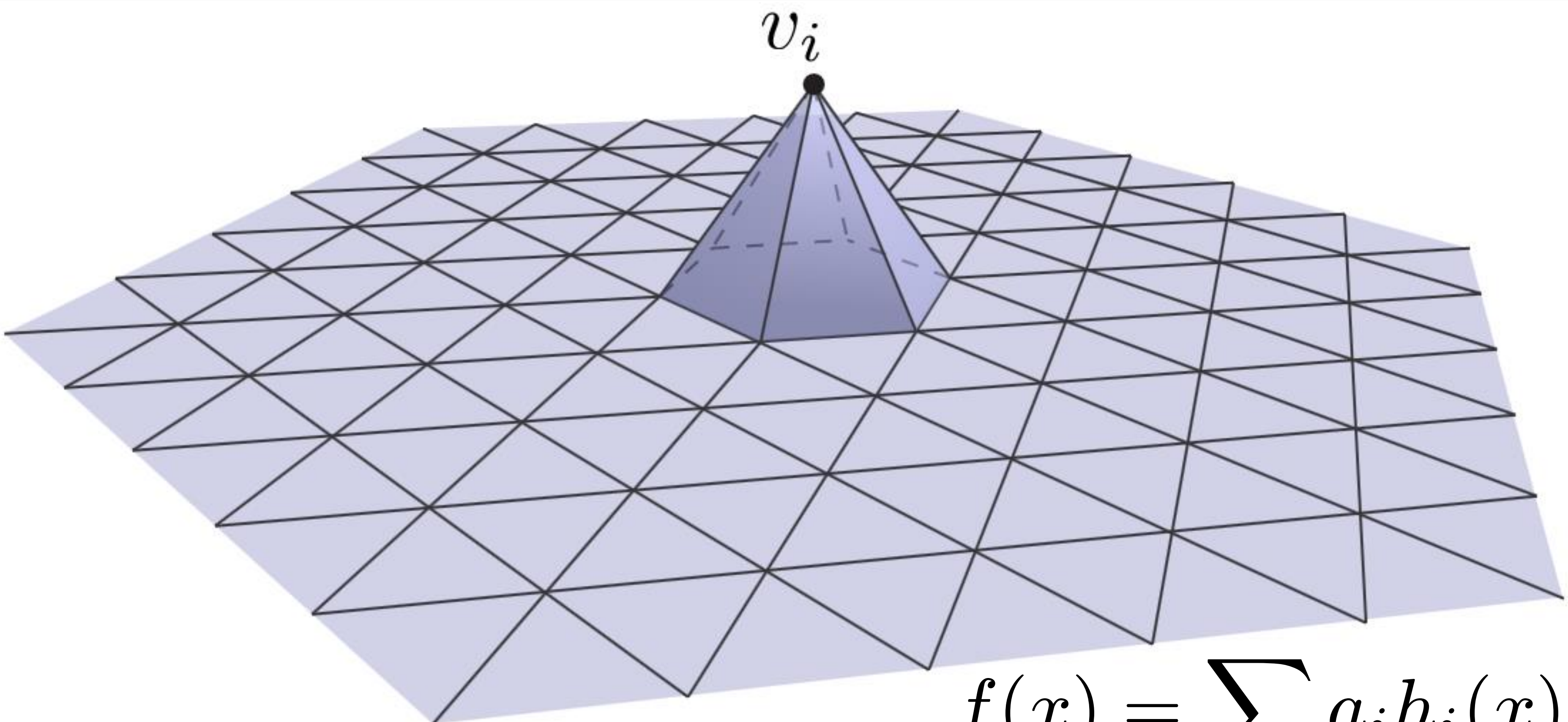
First Order Finite Elements



http://brickisland.net/cs177/wp-content/uploads/2011/11/ddg_hat_function.svg

One "hat function" per vertex

Representing Functions



$$f(x) = \sum_i a_i h_i(x)$$

$$\vec{a} \in \mathbb{R}^{|V|}$$

What Do We Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \; dA$$

Linear combination of hats
(piecewise linear)

What Do We Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \; dA$$

One vector per face

What Do We Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \; dA$$

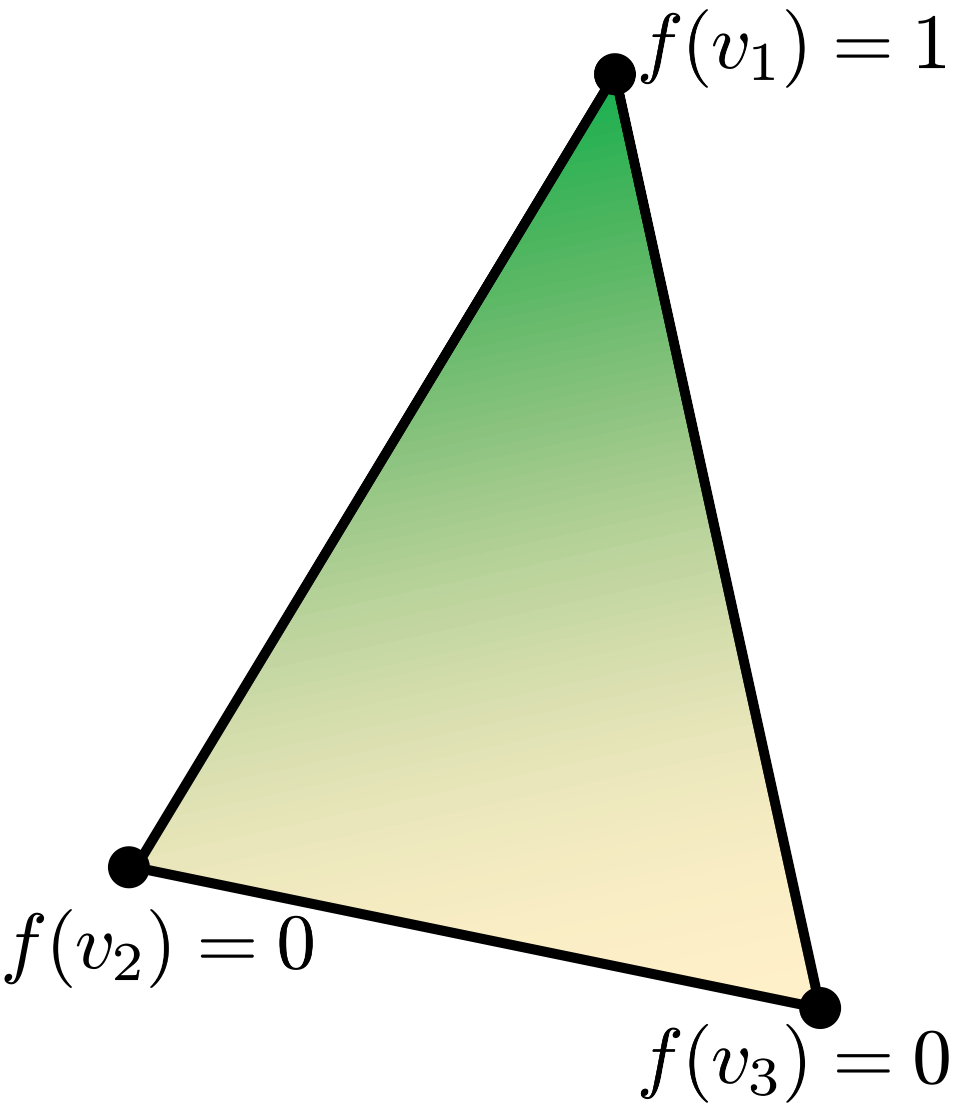
One scalar per face

What Do We Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \nabla g \cdot \nabla f \; dA$$

Sum scalars per face multiplied by face areas

Evaluating the Gradient



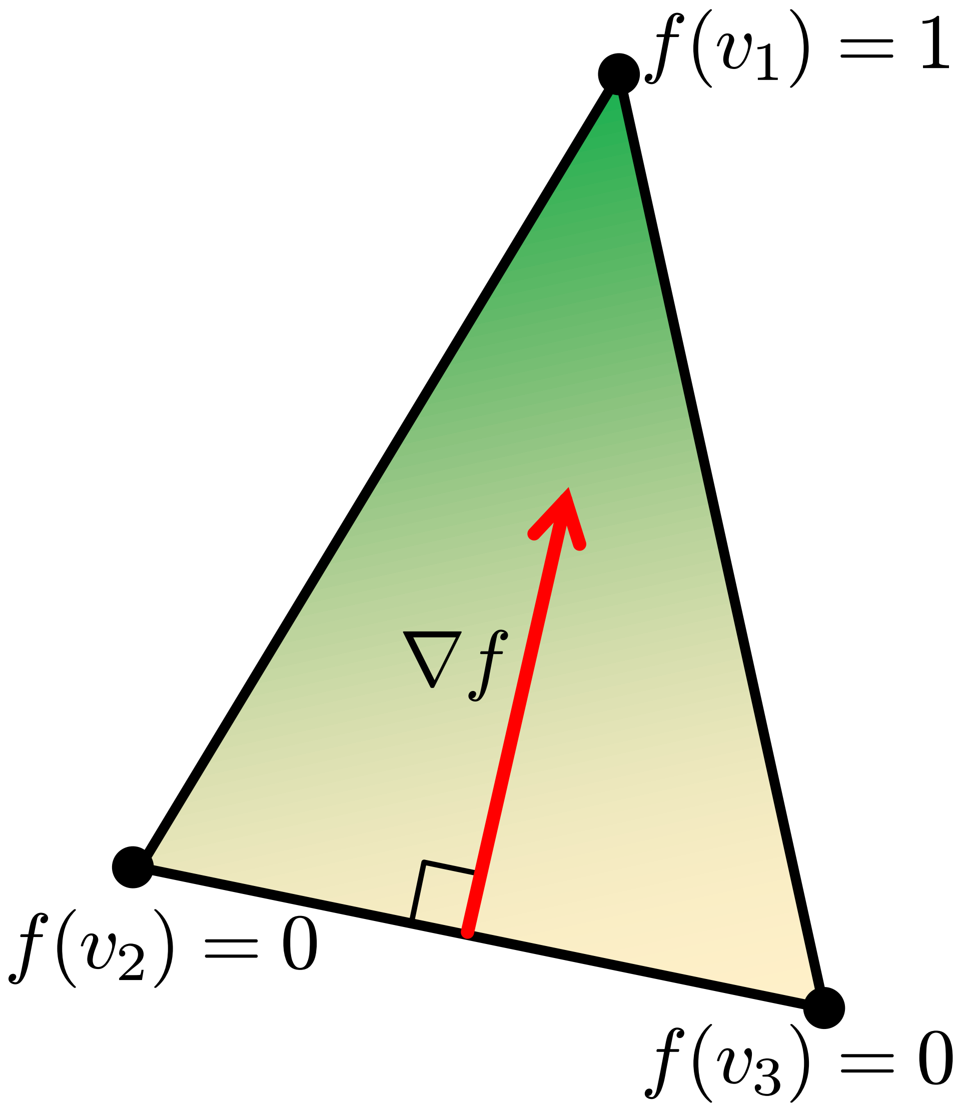
Linear along edges

$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

Evaluating the Gradient



Linear along edges

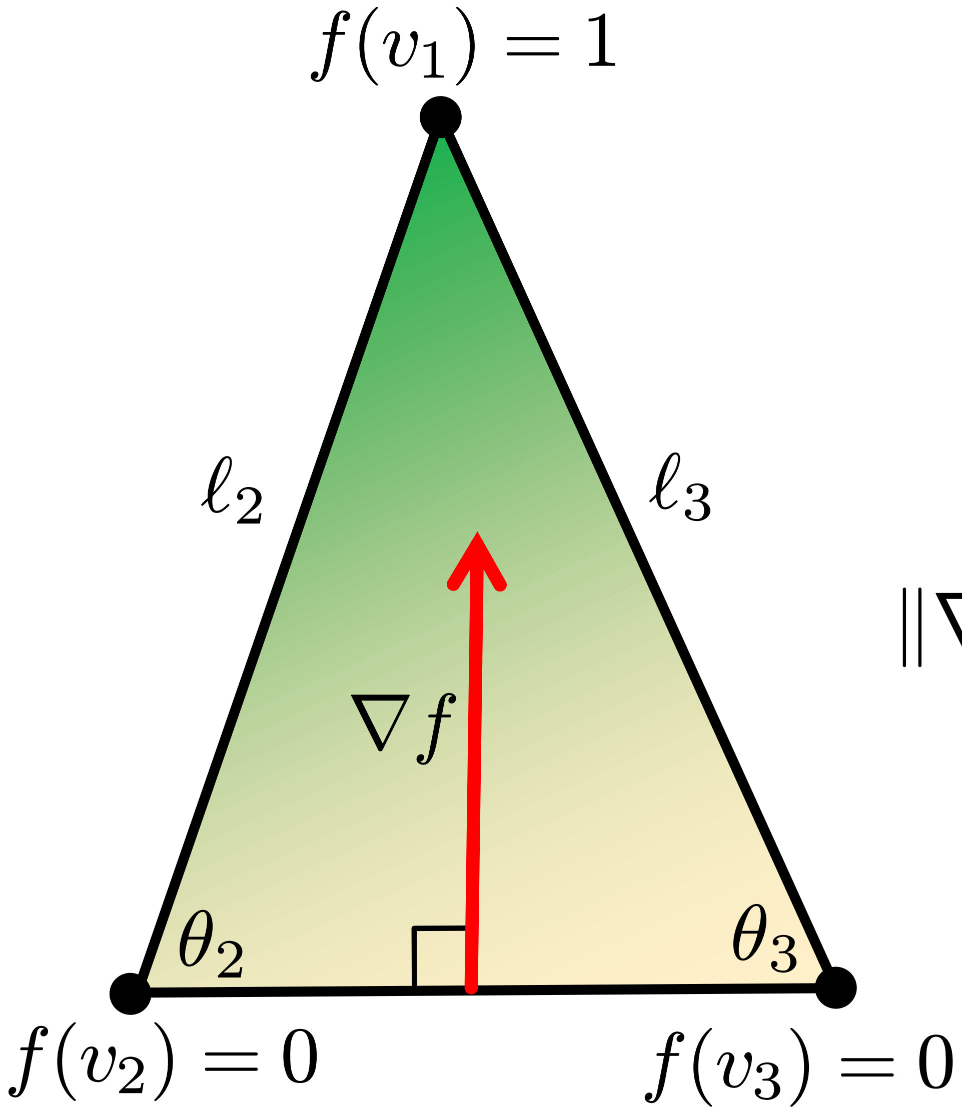
$$\nabla f \cdot (v_1 - v_3) = 1$$

$$\nabla f \cdot (v_1 - v_2) = 1$$

$$\nabla f \cdot n = 0$$

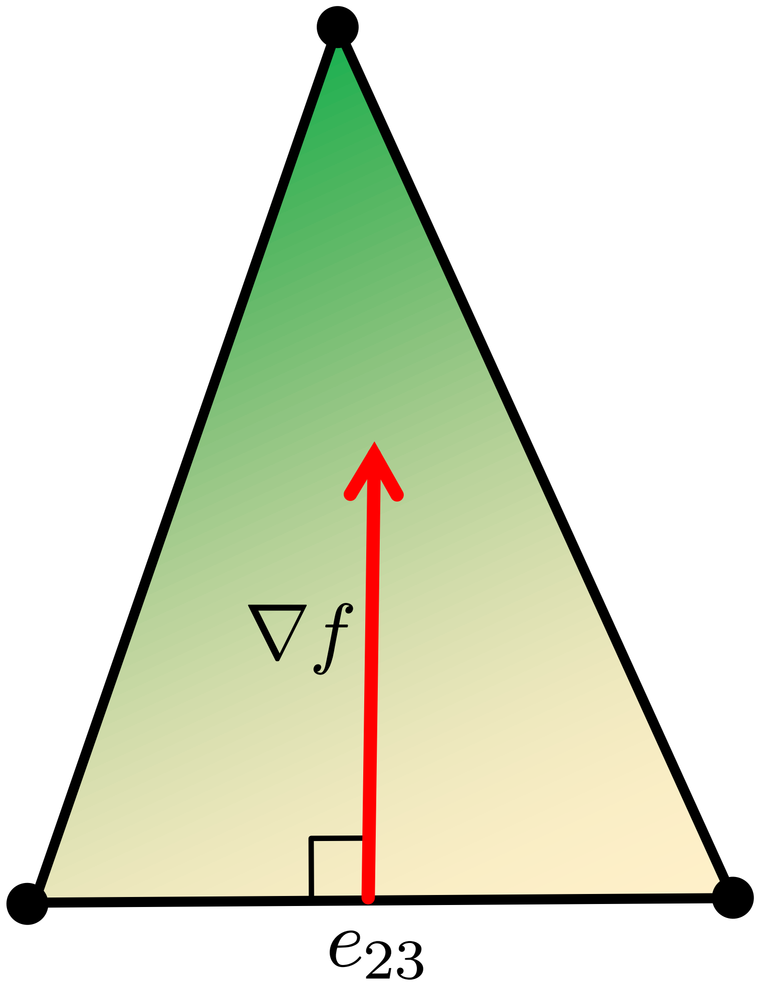
$$\nabla f \cdot (v_2 - v_3) = 0$$

Evaluating the Gradient



$$
\begin{aligned}
1 &= \nabla f \cdot (v_1 - v_3) \\
&= \|\nabla f\| \ell_3 \cos\left(\frac{\pi}{2} - \theta_3\right) \\
&= \|\nabla f\| \ell_3 \sin\theta_3 \\
\|\nabla f\| &= \frac{1}{\ell_3 \sin\theta_3} = \frac{1}{h}
\end{aligned}
$$

Evaluating the Gradient

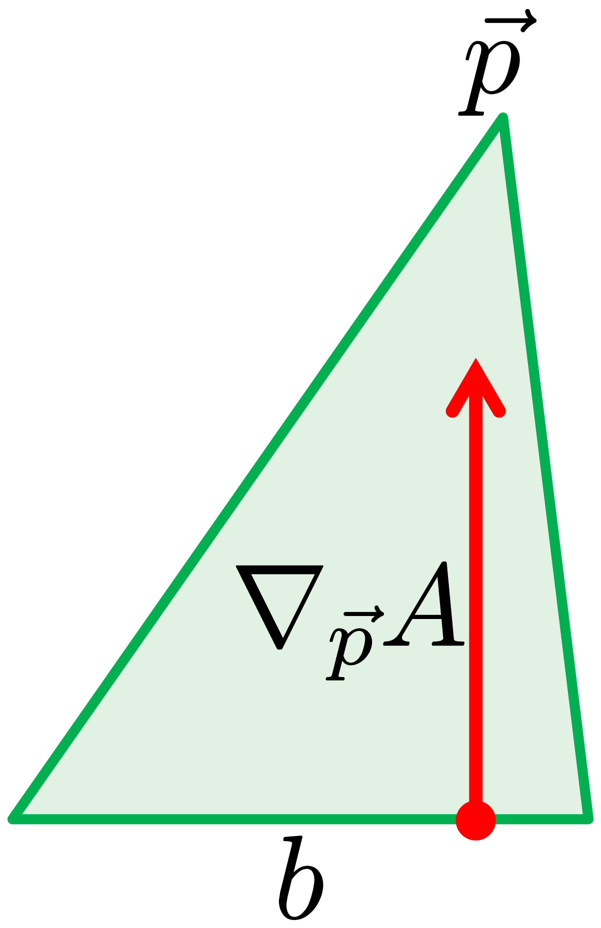


$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

Length of e_{23} cancels "base" in A

Recall:

Single Triangle: Complete



$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_\perp \vec{e}_\perp$$

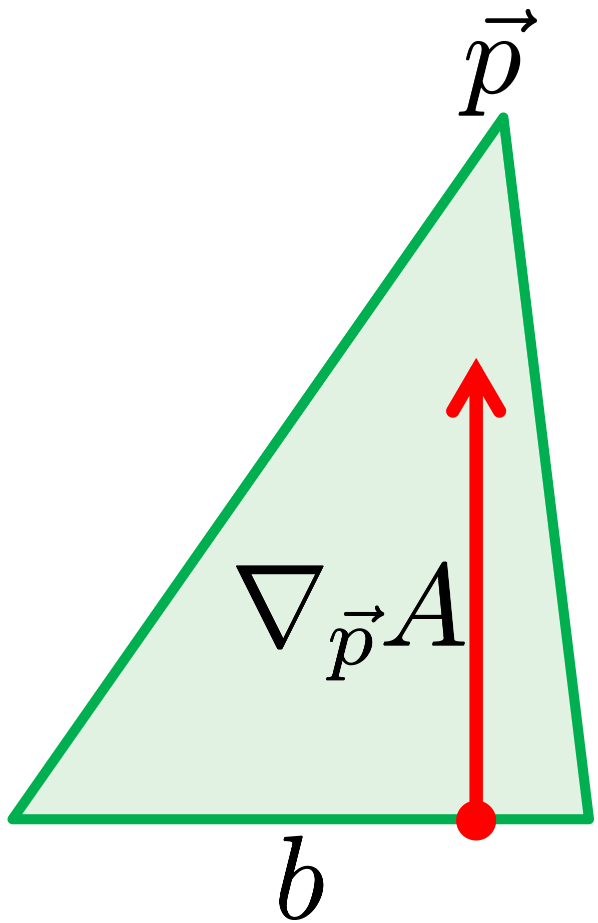
$$A = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_\perp$$

Similar expression

Recall:

Single Triangle: Complete



$$\vec{p} = p_n \vec{n} + p_e \vec{e} + p_\perp \vec{e}_\perp$$

$$A = \frac{1}{2} b \sqrt{p_n^2 + p_\perp^2}$$

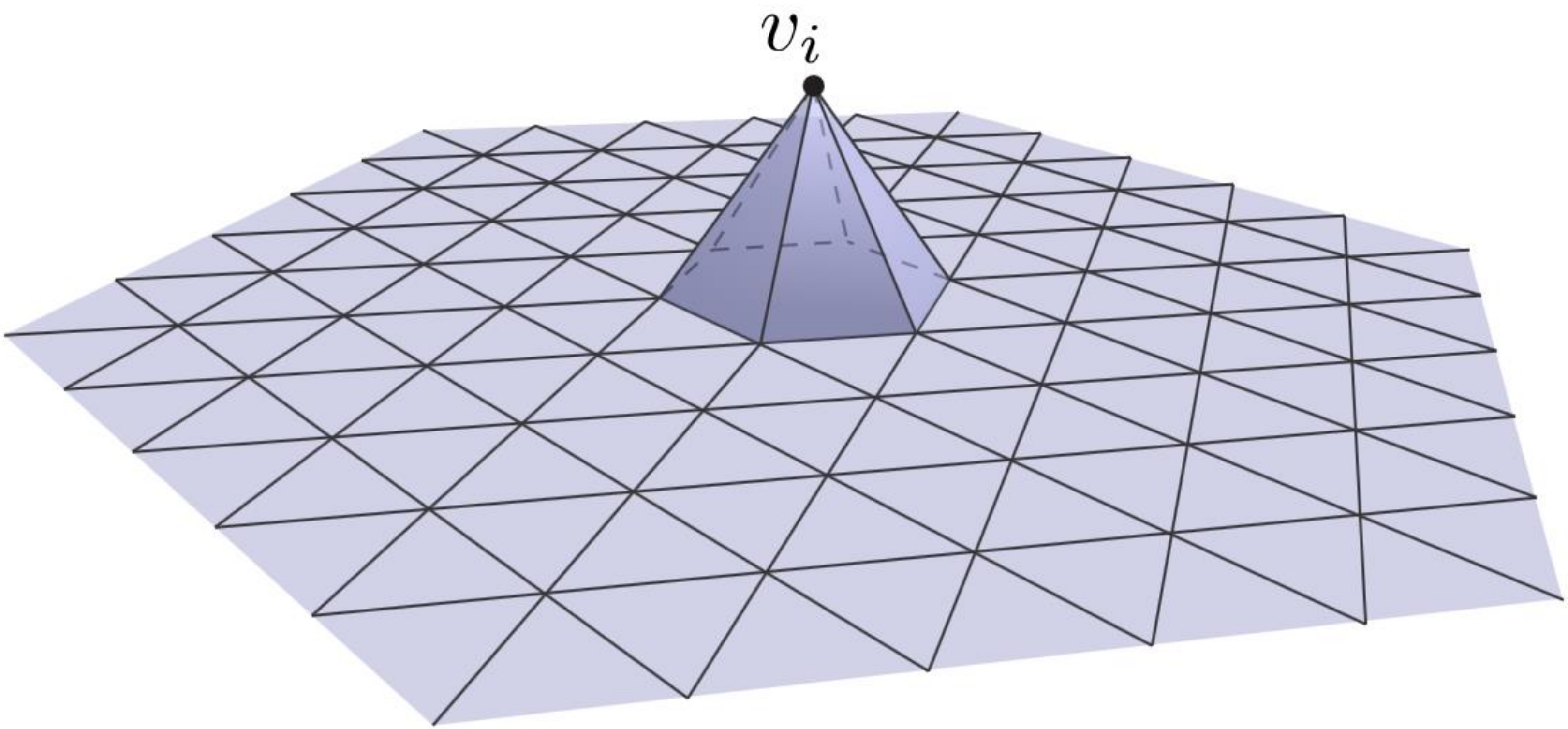
$$\nabla_{\vec{p}} A = \frac{1}{2} b \vec{e}_\perp$$

$$\nabla f = \frac{e_{23}^\perp}{2A} = \frac{\vec{e}_\perp}{h} = \frac{\nabla_{\vec{p}} A}{A}$$

Similar expression

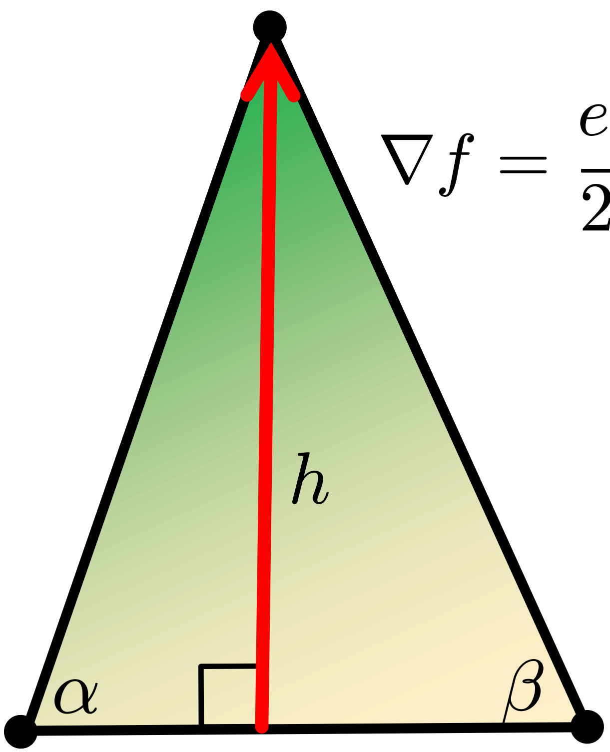
What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \boxed{\nabla g \cdot \nabla f} \, dA$$



What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \boxed{\nabla g \cdot \nabla f}\, dA$$



$$\nabla f = \frac{e_{23}^{\perp}}{2A}$$

Case 1: Same vertex

$$\int_T \langle \nabla f, \nabla f \rangle = A\|\nabla f\|^2$$

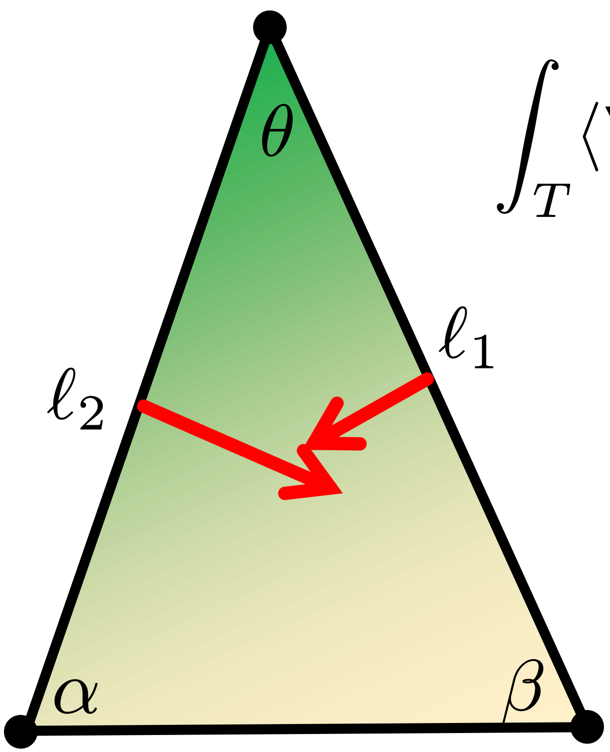
$$= \frac{A}{h^2} = \frac{b}{2h}$$

$$= \frac{1}{2}(\cot\alpha + \cot\beta)$$

What We Actually Need

$$\mathcal{L}_{\Delta f}[g] = -\int_M \boxed{\nabla g \cdot \nabla f}\, dA$$

Case 2: Different vertices



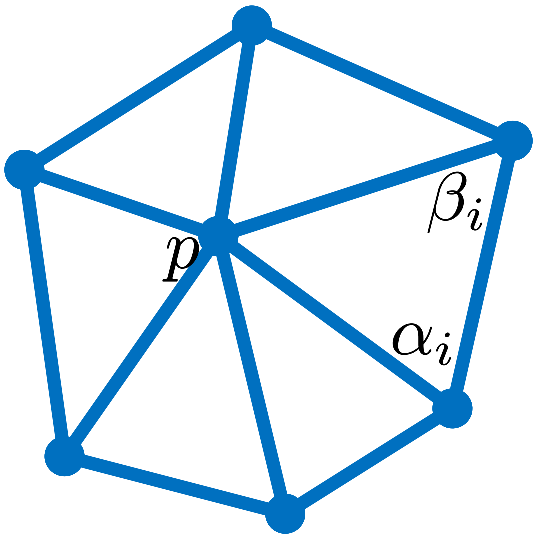
$$\int_T \langle \nabla f_\alpha, \nabla f_\beta \rangle = A \langle \nabla f_\alpha, \nabla f_\beta \rangle$$

$$= \frac{1}{4A} \langle e_{31}^\perp, e_{12}^\perp \rangle = \frac{-\ell_1 \ell_2 \cos\theta}{4A}$$

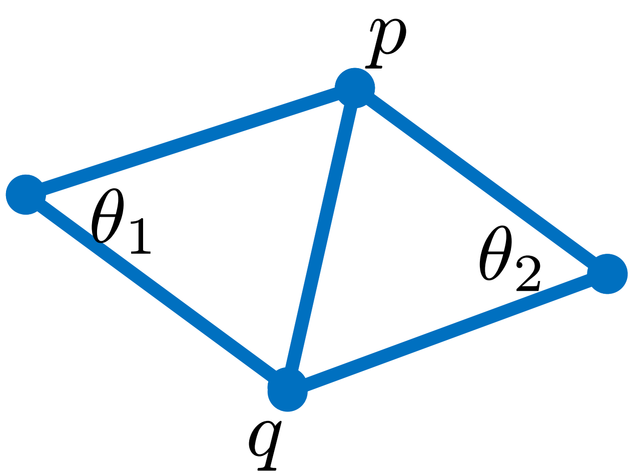
$$= \frac{-h^2 \cos\theta}{4A \sin\alpha \sin\beta} = \frac{-h \cos\theta}{2b \sin\alpha \sin\beta}$$

$$= -\frac{1}{2} \frac{\cos\theta}{\sin(\alpha+\beta)} = -\frac{1}{2} \cot\theta$$

Summing Around a Vertex



$$\langle \nabla h_p, \nabla h_p \rangle = \frac{1}{2} \sum_i (\cot \alpha_i + \cot \beta_i)$$



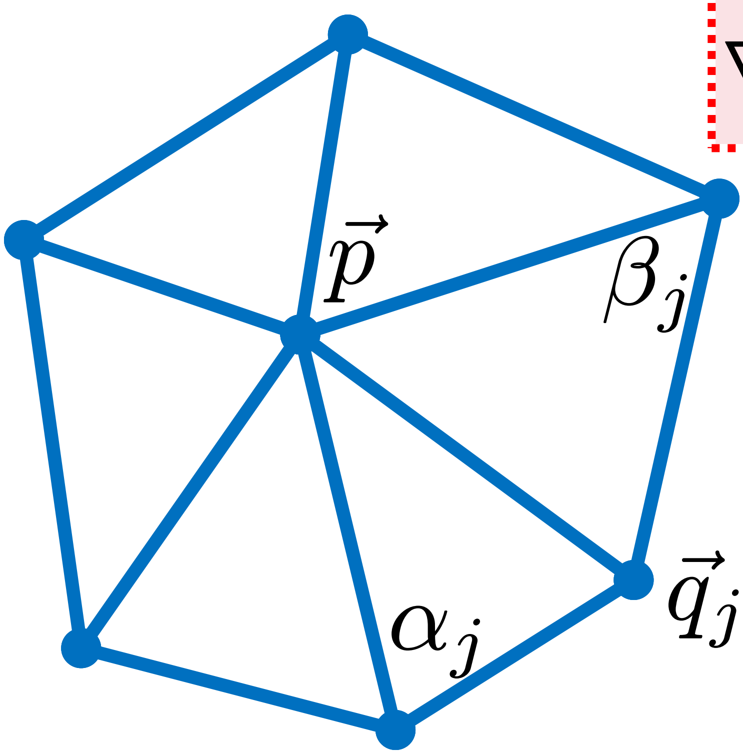
$$\langle \nabla h_p, \nabla h_q \rangle = -\frac{1}{2} (\cot \theta_1 + \cot \theta_2)$$

Recall:

Summing Around a Vertex

$$\nabla_{\vec{p}} A = \frac{1}{2} \sum_j (\cot \alpha_j + \cot \beta_j)(\vec{p} - \vec{q}_j)$$

$$\nabla_{\vec{p}} A = \frac{1}{2}((\vec{p} - \vec{r}) \cot \alpha + (\vec{p} - \vec{q}) \cot \beta)$$



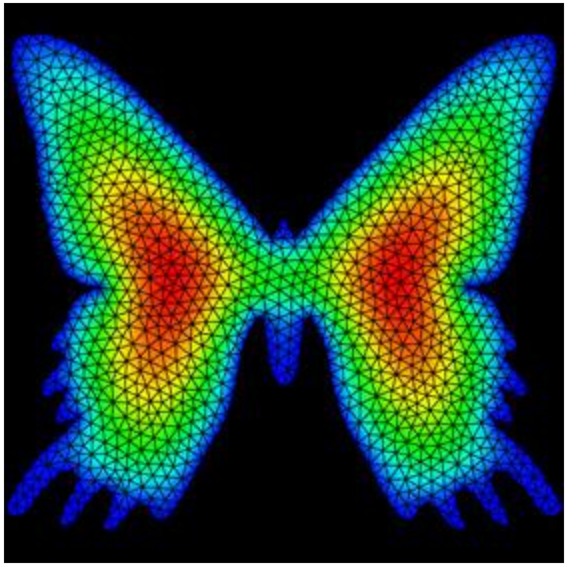
Same weights up to sign!

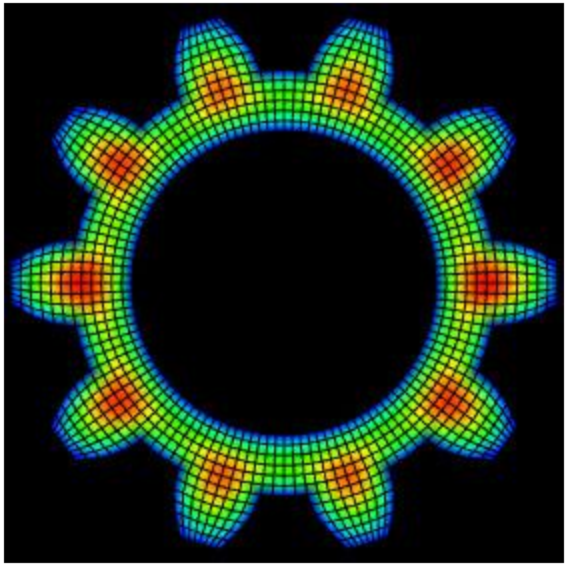
The Cotangent Laplacian

$$L_{ij} = \begin{cases} \frac{1}{2}\sum_{i\sim j}(\cot\alpha_j + \cot\beta_j) & \text{if } i = j \\ -\frac{1}{2}(\cot\alpha_j + \cot\beta_j) & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Poisson Equation

$$\Delta f = g$$

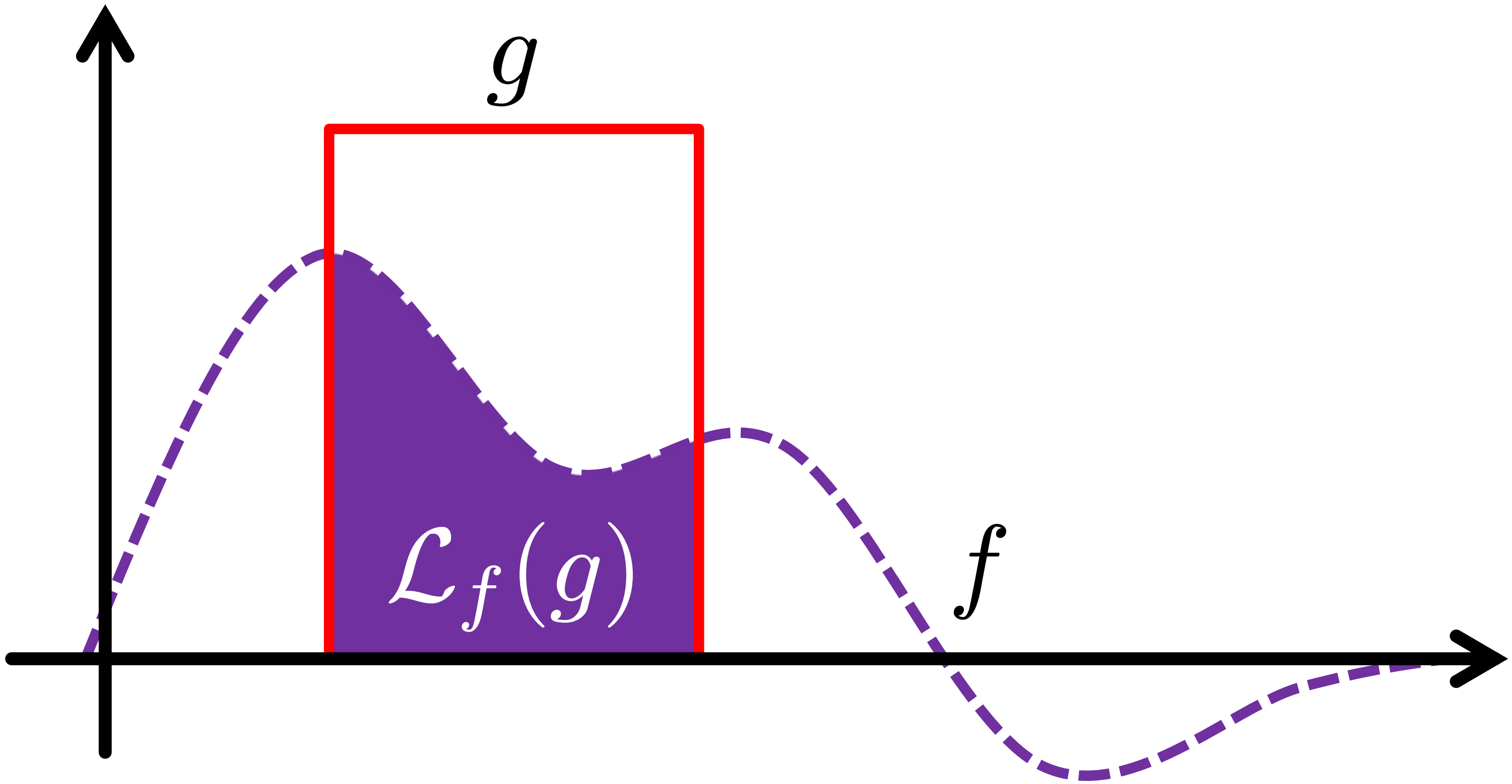




http://nylander.wordpress.com/2006/05/24/finite-element-method-fem-solution-to-poisson%E2%80%99s-equation-on-triangular-mesh/

Weak Solutions

$$\int_M \phi \Delta f \, dA = \int_M \phi g \, dA \; \forall \text{ test functions } \phi$$



Finite Elements Weak Solutions

$$\int_M h_i \Delta f \; dA = \int_M h_i g \; dA \; \forall \text{ hat functions } h_i$$

$$\begin{aligned}
\int_M h_i \Delta f \; dA &= -\int_M \nabla h_i \cdot \nabla f \; dA \\
&= -\int_M \nabla h_i \cdot \nabla \sum_j a_j h_j \; dA \\
&= -\sum_j a_j \int_M \nabla h_i \cdot \nabla h_j \; dA \\
&= \sum_j L_{ij} a_j
\end{aligned}$$

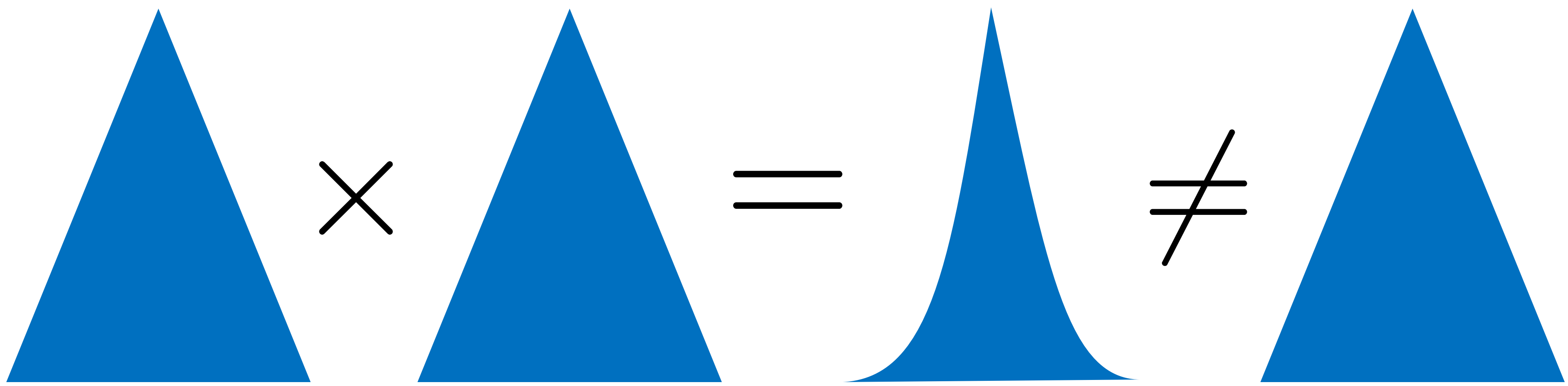
Stacking Integrated Products

$$\begin{pmatrix} \int_M h_1 \Delta f \; dA \\ \int_M h_2 \Delta f \; dA \\ \vdots \\ \int_M h_{|V|} \Delta f \; dA \end{pmatrix} = \begin{pmatrix} \sum_j L_{1j} a_j \\ \sum_j L_{2j} a_j \\ \vdots \\ \sum_j L_{|V|j} a_j \end{pmatrix} = L\vec{a}$$

Multiply by Laplacian matrix!

Problematic Right Hand Side

$$\int_M h_i \Delta f \; dA = \int_M h_i g \; dA \;\; \forall \text{ hat functions } h_i$$



Product of hats is quadratic

A Few Ways Out

- Just do the integral
  - “Consistent” approach
- Approximate some more
- Redefine *g*

A Few Ways Out

- Just do the integral
  “Consistent” approach
- Approximate some more
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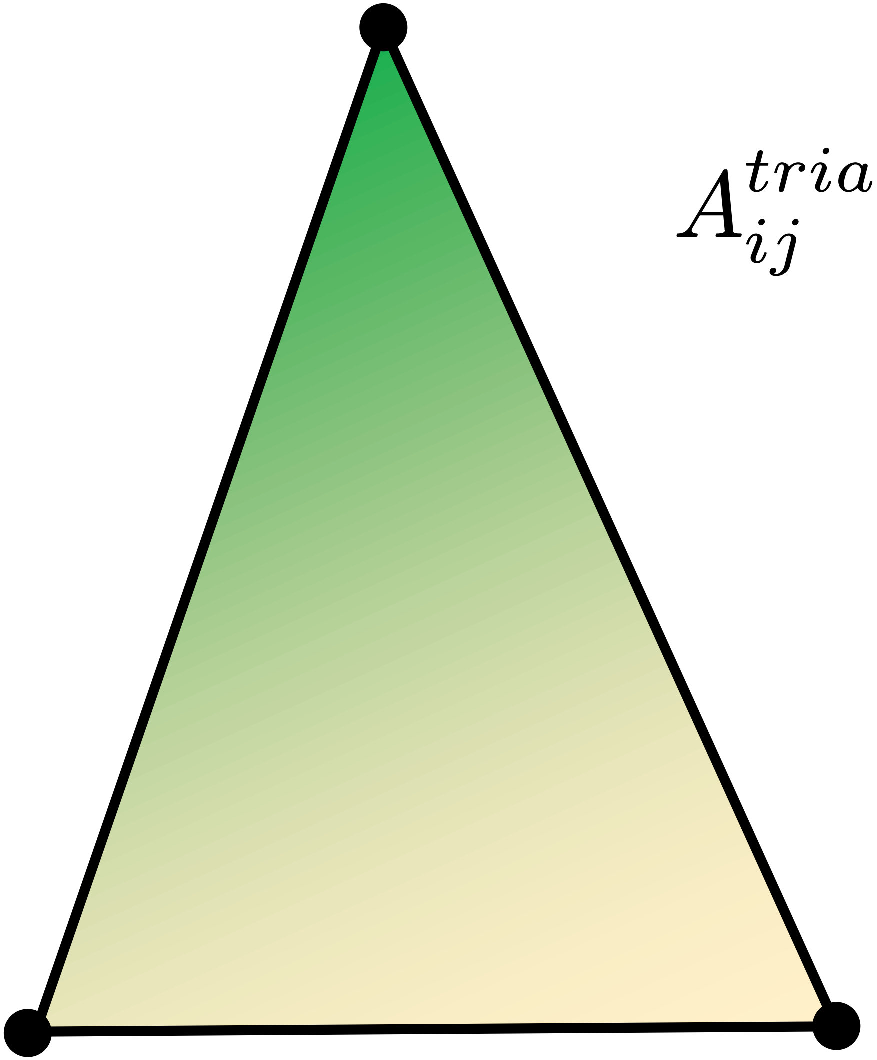
The Mass Matrix

$$A_{ij} = \int_M h_i h_j \, dA$$

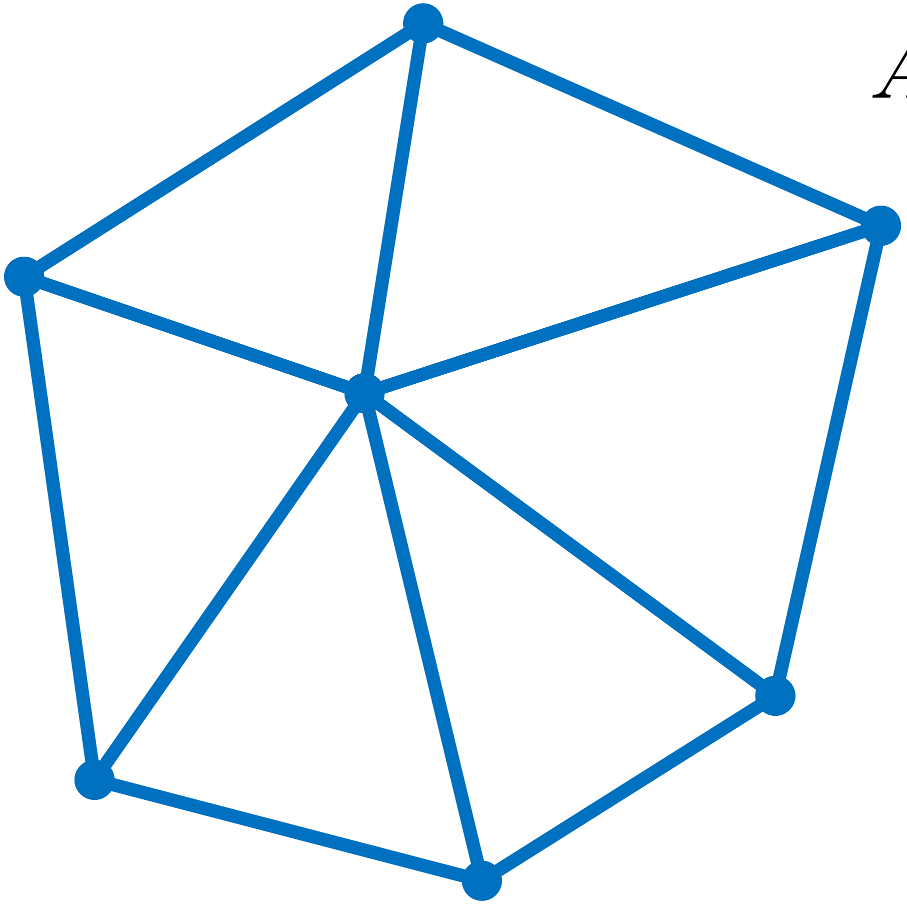
- **Diagonal elements:** Norm of h_i
- **Off-diagonal elements:** Overlap between h_i and h_j

Consistent Mass Matrix



$$A_{ij}^{triangle} = \begin{cases} \frac{\text{area}}{6} & \text{if } i = j \\ \frac{\text{area}}{12} & \text{if } i \neq j \end{cases}$$

Summing Around Triangles



$$A_{ij} = \begin{cases} \frac{\text{one-ring area}}{6} & \text{if } i = j \\ \frac{\text{adjacent area}}{12} & \text{if } i \neq j \end{cases}$$

Properties of Mass Matrix

- Rows sum to one ring area / 3
- Involves only vertex and its neighbors
- Partitions surface area

Non-diagonal matrices aren't fun.

Use for Integration

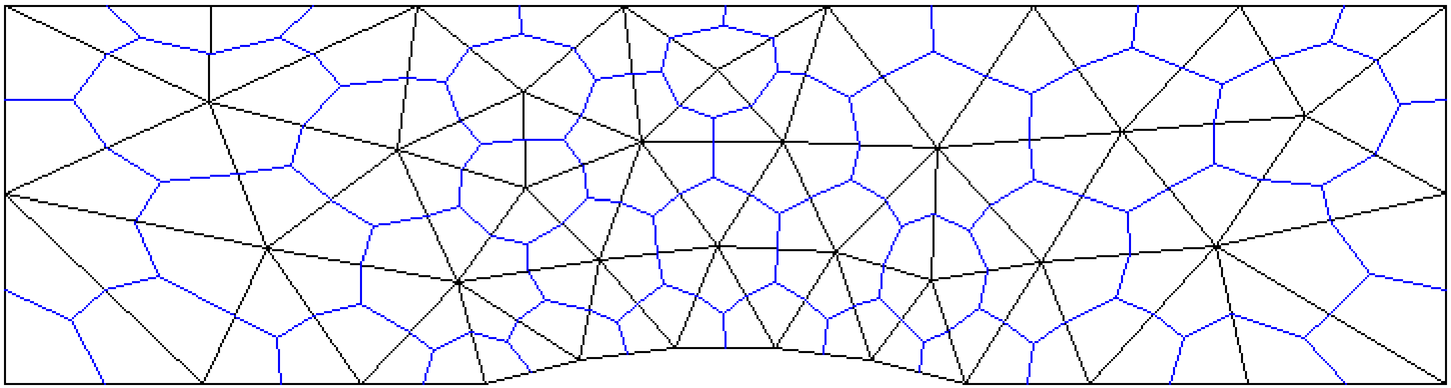
$$\int_M f = \int_M \sum_j a_j h_j$$

$$= \int_M \sum_j a_j h_j \sum_i h_i$$

$$= \sum_{ij} A_{ij} a_j$$

$$= \mathbf{1}^\top A \vec{a}$$

Lumped Mass Matrix



$$\tilde{a}_{ii} = \text{Area}(\text{cell } i)$$

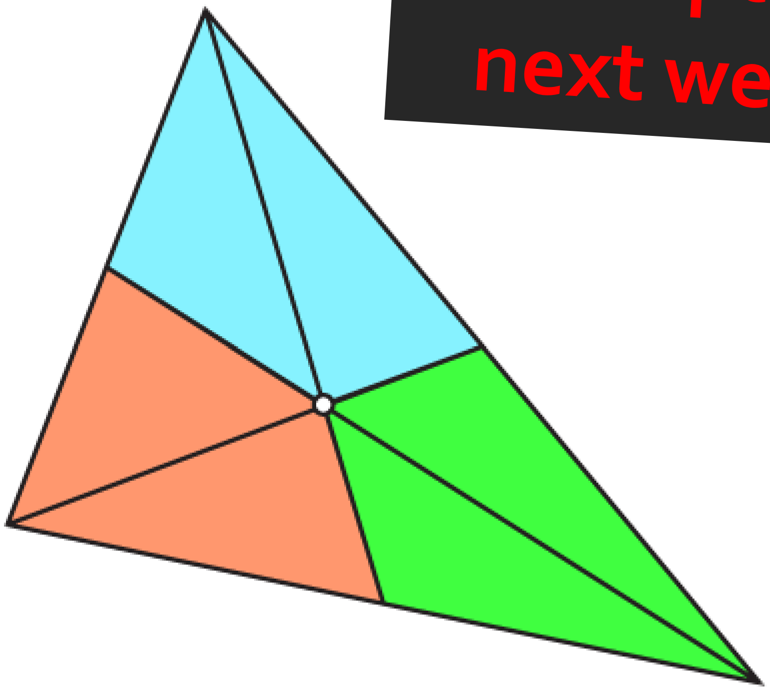
Won't make big difference for smooth functions

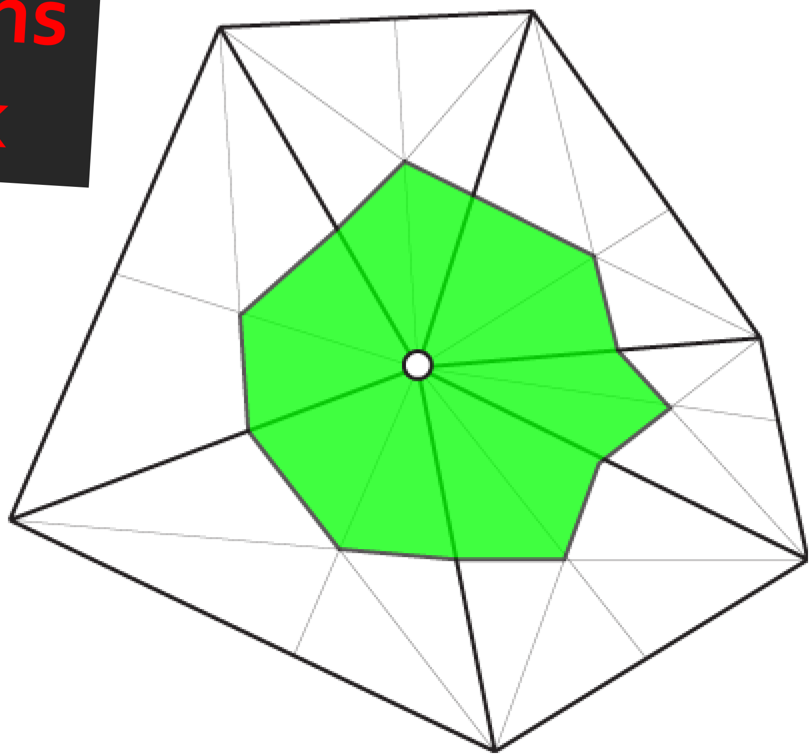
http://users.led-inc.eu/~phk/mesh-dualmesh.html

Approximate with diagonal matrix

Barycentric Lumped Mass

More options next week





http://www.alecjacobson.com/weblog/?p=1146

Area/3 to each vertex

Ingredients

- **Cotangent Laplacian *L***

  Per-vertex function to integral of its Laplacian against each hat

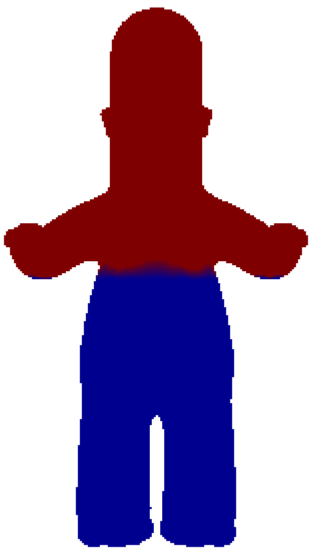
- **Area weights *A***

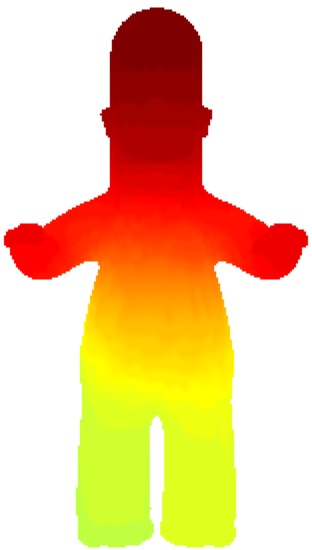
  Integrals of pairwise products of hats (or approximation thereof)

Solving the Poisson Equation

$$\Delta f = g \quad \longrightarrow \quad L\vec{f} = A\vec{g}$$

g



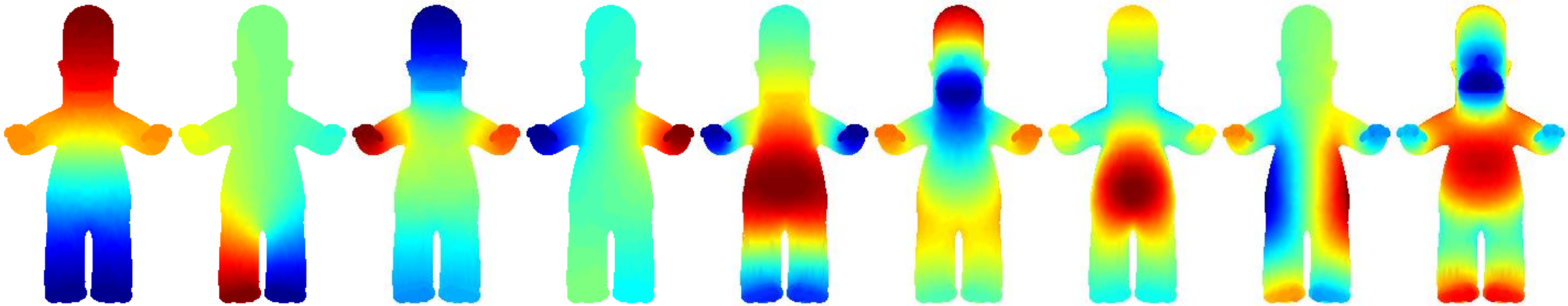


f

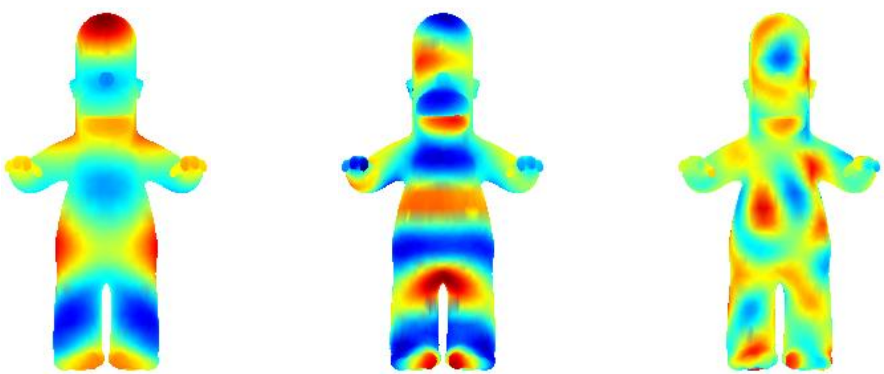
Must integrate to zero

Determined up to constant

Eigenhomers



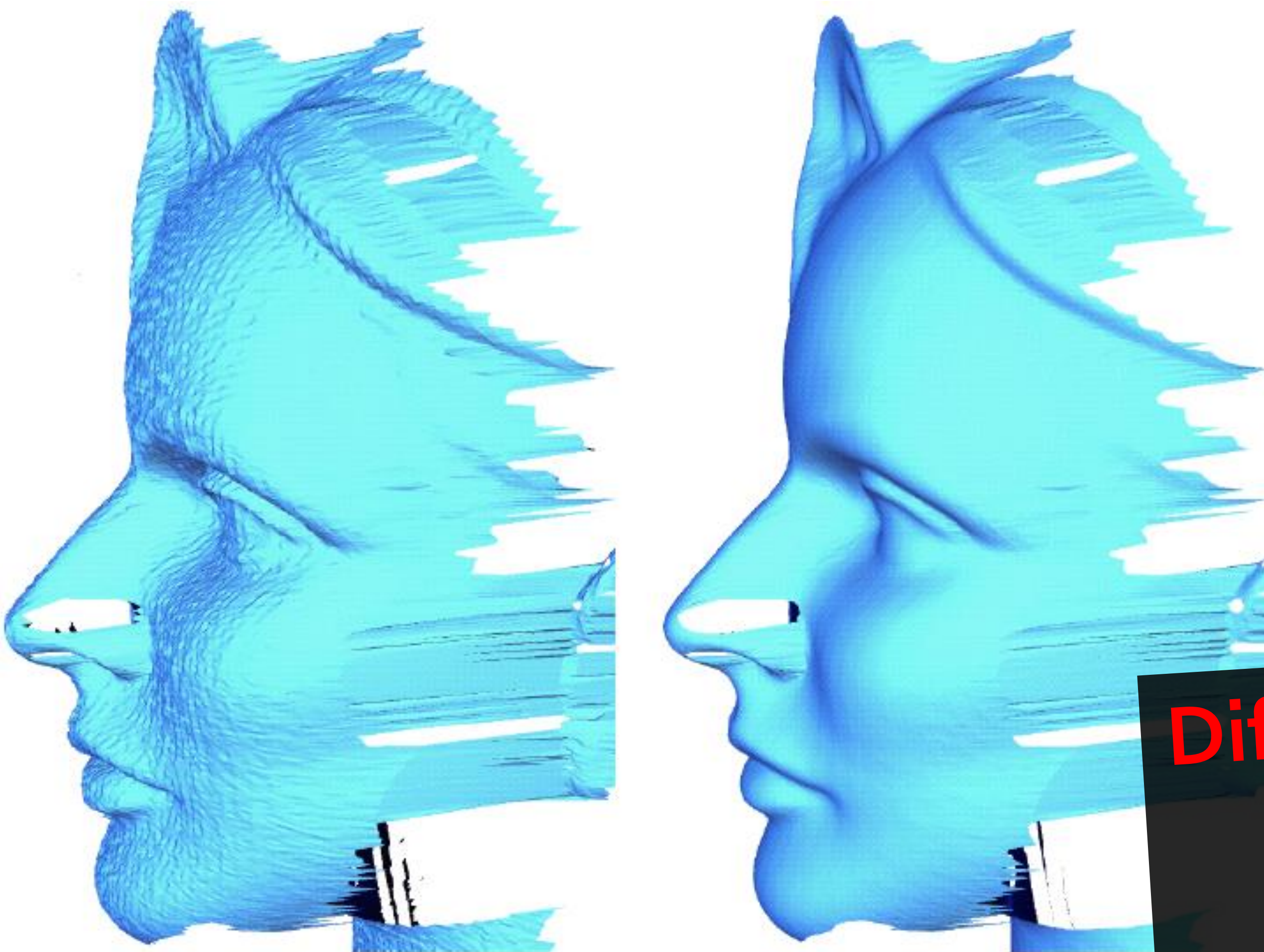
2 3 4 5 6 7 8 9 10



25 50 100

What is smallest eigenvalue?

Implicit Fairing



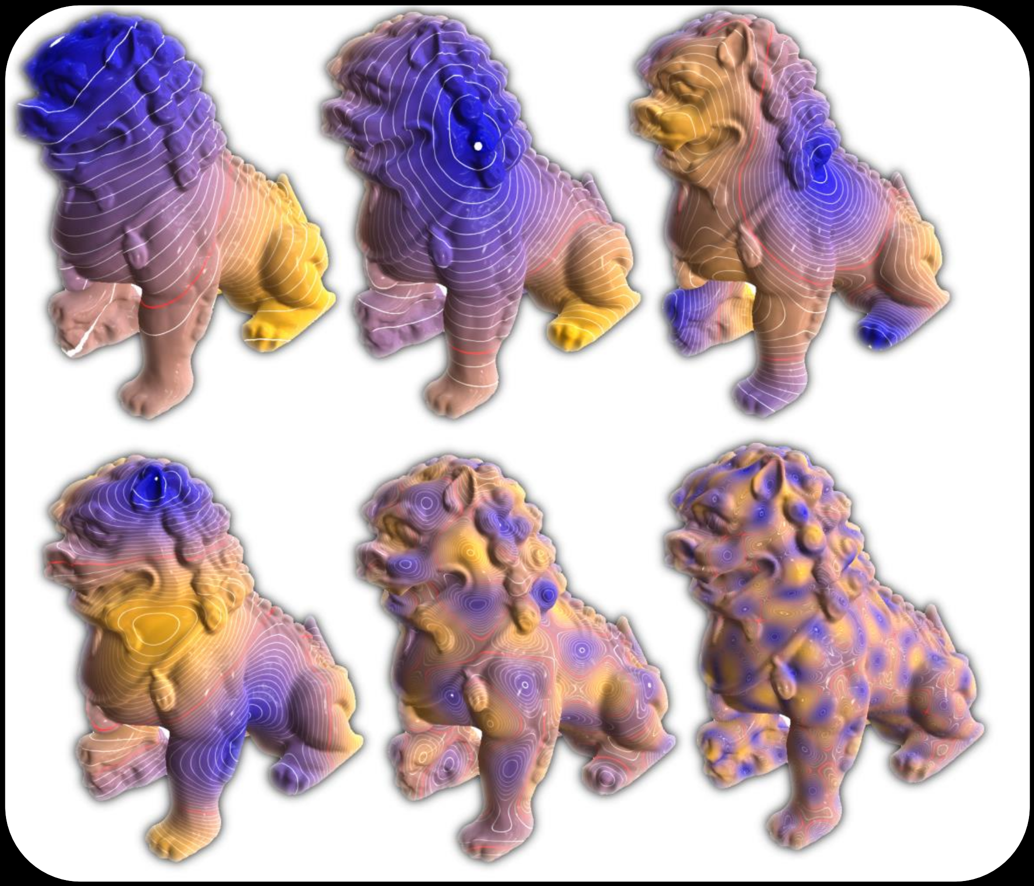
(a) (b)

Diffusion applied to vertex positions

http://multires.caltech.edu/pubs/ImplicitFairing.pdf

Implicit Fairing

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png



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